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SEPTEMBER 2012

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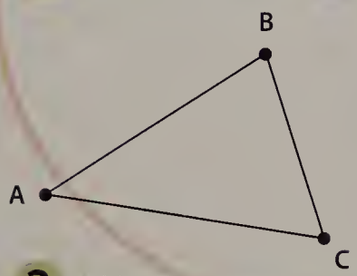

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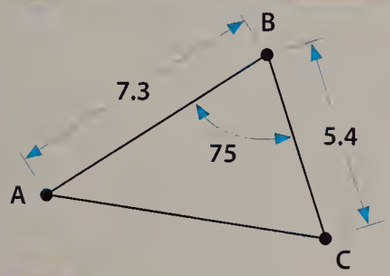

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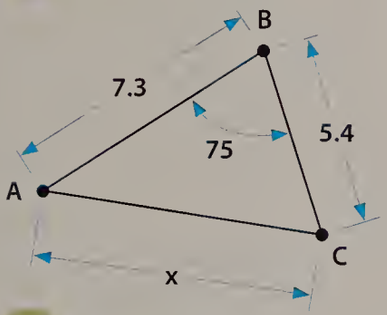

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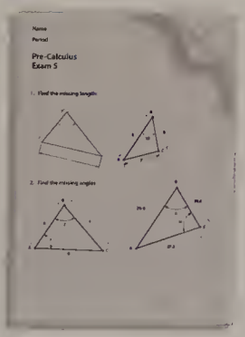

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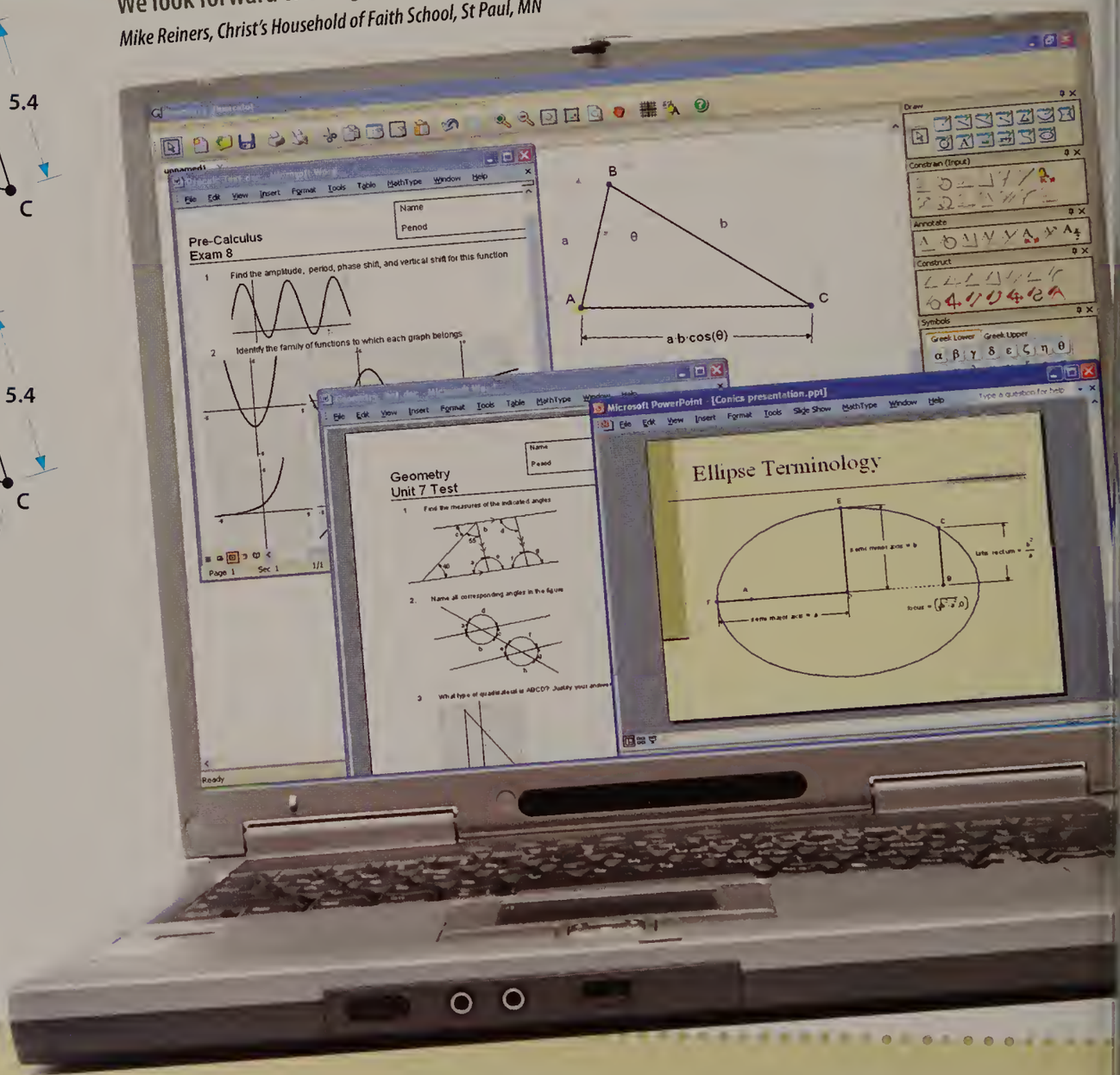

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If you think that the cost of three-dimensional printing prohibits its use in the high school mathematics classroom, Adam Poetzel and colleagues will dispel this thought. Their article "Three-Dimensional Printing: A Journey in Visualization" (pp. 102–7) introduces an affordable way to unveil your students' creativity. The authors give a step-by-step description of how their precalculus and calculus students rotate planar curves around an axis and make three-dimensional objects from them.

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- Mathematical Lens (p. 96): An additional photograph for your own mathematical questions

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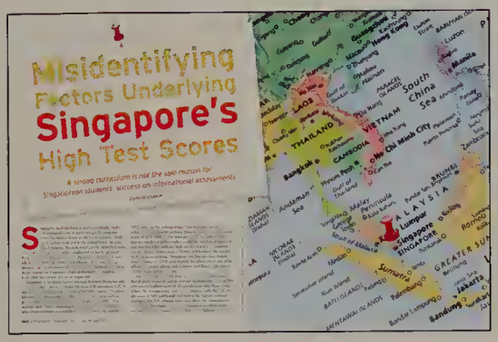

FOLLOW-UP TO "MISIDENTIFYING FACTORS UNDERLYING SINGAPORE'S HIGH TEST SCORES"

Perspective from a Vocational School

Zalman Usiskin's article "Misidentifying Factors Underlying Singapore's High Test Scores" (*MT* May 2012, vol. 105, no. 9, pp. 666–70) was enlightening. I always suspect that when the United States is compared with other countries, we are not comparing apples with apples.

When I visited Austria a few years ago, I learned that after middle school, students are directed to either a vocational training track or a college track. What I wasn't sure about is whether the vocational students are tested along with the college-bound students—as ours are.

I teach at a regional vocational high school in Connecticut. Our high-stakes test scores tend to be lower than those in the feeder districts because high schools in the region often encourage lower-academic, non-college-bound students to enroll in my school.

Anna Roode
adroode@yahoo.com
Henry Abbott Technical High School
Danbury, CT, May 5, 2012

Getting the Full Story

When I received my May 2012 *Mathematics Teacher*, I looked through the table of contents, found Usiskin's article, turned to it immediately, and began to read. I sat up and exclaimed, "Finally! Someone who is giving the full story about the comparison of U.S. test scores with those of Singapore and other high-performing nations we are compared with so frequently today."

With my more than thirty-four years of experience teaching mathematics in U.S. public schools, in both non-affluent and affluent areas, I know that much occurs outside the school day that greatly affects the learning and scoring of our students. I am proud that the United States values the education of *all* students. I agree completely with Usiskin's statement that "virtually all students can learn mathematics to a high level if they receive a well-designed curriculum with high expectations and devote the time, both in school and out of school, that it takes to learn such a curriculum."

We would serve our country and our youth better if we worked toward improving the circumstances of our students and their families—circumstances that so greatly impact their learning—and less time comparing them with Singaporean students or others. The select populations that other countries test and the populations that we test are not the same and should not be compared as if they were.

Thanks to Dr. Usiskin for this article!

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Mountain Brook, AL, May 7, 2012

Is Anyone Listening?

I appreciate Usiskin's excellent article. Sadly, I don't think that our policymakers will pay it any attention. Here in Buffalo, teachers are fighting a state-mandated school evaluation policy that does not take into account student absence, a critical problem in urban schools. The teachers are under extraordinary pressure from greedy politicians and the local newspaper to cave in, because the state will punish the school system if it does not adopt this policy. The state's message: Sign on, or we'll withhold millions of dollars. This is what nowadays is called negotiation.

For some years now, I have felt very happy to be getting old. We lived in better times.

Gerald R. Rising
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SUNY Distinguished Teaching Professor
University at Buffalo
Buffalo, NY, May 8, 2012

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged, but because of the large number submitted, we do not send letters of acceptance or rejection. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

Picking apart an Argument, Point by Point

Usiskin cites four noncurricular—in fact, nonschool—factors that contribute to the high performance of Singaporean students on international comparative exams: (1) Singapore is very affluent, and the low-paying jobs are done by foreigners, whose children are not being measured by the exams; (2) Singapore administers a high-stakes exam (the PSLE) at the end of sixth grade; (3) Singaporean students are less distracted by sports, social life, and other activities that loom large in American culture; and (4) Singaporean students get additional tutoring outside school.

These statements are all true, and these factors probably do contribute to the impressive performance of Singaporean students on the exams. However, Usiskin's discussion needs to be supplemented by two further points:

- Factors 2, 3, and 4 are all aspects of the general proposition that Singapore, as a nation, places much higher value on education than does the United States.
- Factor 1 is evidence that Singapore's high esteem of education has paid off handsomely for the country.

Usiskin's discussion of the first factor is naive in that it does not take into account the rapid growth of Singapore's wealth. Singapore has existed as an independent country for fewer than fifty years. When it achieved independence from Malaysia, it was no richer than the surrounding countries. Its current affluence is the result of a massive effort, and one main lever enabling its rise has been the emphasis on education. In the mid-1990s, when Singapore first achieved recognition by coming out at the top of the international exam results, it was considerably less affluent than it is now and had fewer foreign workers. So in emphasizing factor 1, and especially in placing it first, Usiskin is in a sense penalizing Singapore for its own success, significantly enabled by the high value it places on education. Singapore has transformed itself from one of the poorer nations in the world to outstripping the United States on a per capita basis. Taking into account the time development behind Usiskin's first point is a powerful argument for the importance of education, something that he tries to downplay in his closing remarks.

Also, we should keep in mind that the Singaporean students who are being examined are all the Singaporean students, including children of the people who did low-end work a generation ago, when Singapore was not rich enough to import help.

To some extent, factor 4 is also an artifact of the economic success. Desire for success in school has been fairly constant in Singapore, but affluence allows it to be expressed in new ways, such as by hiring tutors. A cursory tour of a bookstore in one of Singapore's Housing and Development Board developments reveals that approximately half the inventory is devoted to supplementary books for extra study, with mathematics a prominent subject. With or without outside tutoring, Singaporeans have been hitting the schoolbooks for a long time. Again, this effort has served them well.

Also regarding factor 4, in the mid-1990s, when the original TIMSS was conducted, additional information was gathered about Germany, Japan, and the United States. Those data confirmed Usiskin's point that Japanese students spend time supplementing school instruction with private tutors (*juku*). However, the hours of instruction per middle school student was about the same in the United States and Japan (145 hours per year) and much less in Germany (115 hours). The difference was that in the United States it was all in-school instruction, whereas Japan and Germany had about the same amount of in-school instruction, which Japan then supplemented with the *juku* work. Japan's test results were much higher than either the United States's or Germany's, which were similar, with Germany's somewhat higher. The conclusions I draw are that Germany provided too little instruction in mathematics and that the instruction provided in U.S. schools was comparatively ineffective.

So I would regard Usiskin's factors 2 and 3 as perhaps the ones to ponder the most. As noted, I regard them as being part of the general proposition that Singapore, as a nation, takes education more seriously than does the United States and factor 1 as providing justification for that policy stance. But factors 2 and 3 must be seen as part of the picture that includes a system that recruits highly capable teachers and creates high-quality curricula and textbooks to help them do their job.

Teachers and textbooks are only part of the picture, but in emphasizing factors 2, 3, and 4, Usiskin perhaps moves from his early view of considering only the school system to undervaluing it. The danger in this shift is in the aid and comfort it gives to U.S. policymakers who might be willing to shirk the responsibility for ensuring a highly capable teaching corps through high standards for induction, continued professional development, and incentives—in the form of salaries and other kinds of job satisfaction, including cooperative work to improve, and other professional activities.

A final point: In his penultimate paragraph, Usiskin recites the well-known fact that the United States has had mediocre performance on exams for fifty years but has done well economically. And he faults Japan for faltering despite its high scores. Clearly, many factors affect overall economic performance, education being an important but hardly the only one. I am not alone in believing that the United States has managed to compensate for its mediocre investment in schools by importing talent from around the world. There are many reasons for supposing that this policy has a limited future. As to Japan, its economic troubles started with the overheating of its economy during the 1980s, when its dramatic rise, probably no doubt fueled by strong education, was a topic of grave concern in the United States. Japan's property bubble burst was probably more severe than ours of 2007. It remains to be seen whether we can do a better job than Japan of recovering from such an event.

Roger Howe
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New Haven, CT, May 17, 2012

The Rush to CCSSM

Usiskin writes, "The misidentification of the causes of high scores in Singapore has led to what I believe are unfortunate policy decisions regarding the Common Core State Standards and their implementation" (pp. 669–70). I heartily agree that we need to scrutinize CCSSM far more carefully.

When the first of several NCTM Standards publications appeared in 1989, a brief honeymoon period ended with the start of the "math wars." Groups such as Mathematically Correct and NYC-HOLD became the nexus for vocal and well-organized resistance to NCTM's vision of equitable, high-quality mathematics education as well as to various curriculum projects and texts inspired by NCTM's ideas. The voices opposed to progressive mathematics education have not succeeded in destroying that vision, but they have been able to retard change or undermine and undo it in many districts across the country.

Conceivably, many NCTM members and leaders may believe that the Common Core State Standards for Mathematics (CCSSM) are a godsend that will actualize NCTM's ideals more thoroughly and effectively than did anything in the previous twenty years. And so we hear virtually no criticism or questioning from NCTM or other professional mathematics organizations about CCSSM, how it has come into being, how it is going to be implemented and assessed, or what it means for progressive educational values in general.

This passive acceptance is a grievous mistake. I believe that NCTM's recent and current leadership has been seduced by the education "deform" movement—a coalition of corporate, political, and conservative social forces that seeks to undermine U.S. public education and turn it into a bland, business-friendly private enterprise that will suppress teacher and student creativity and autonomy to maximize profit and minimize resistance and individuality.

CCSSM does include process standards that recall the principles for mathematics education in NCTM's publications from 1989 until 2008, but where the rubber meets the road—namely, in teacher education, classroom practice, and high-stakes assessment—I predict that there will be little or no change from teacher-centered instruction dominated by a rigid set of content standards grounded in a one-size-fits-all approach to individual children.

Reading through CCSSM's content sections leads one to conclude that the authors have ignored the notion of differential student intellectual and emotional growth or attainment. All students are expected to know "piece of content X" by "grade Y," and teachers who fail to get Z percent of their students to show mastery on schedule will fear for their jobs. As for students who might benefit from another mathematical journey, they're out of luck.

The problem is in no small part due to the top-down, corporate-driven nature of CCSSM. We continue to believe—mistakenly—that raised bars and toughened standards, with repeated calls for "accountability" and many sticks with occasional carrots (see Race to the Top, an antidemocratic program no one voted for or dreamed of during the 2008 campaign), will make "lazy" teachers abandon their alleged "soft bigotry of lowered expectations" and usher in a new golden age of mathematics achievement in which we win imaginary international competitions. In these competitions, we are supposed to be trailing according to test scores that don't compare apples with apples, that ignore a host of crucial differences between the United States and countries like Singapore, and that do not examine meaningful mathematical achievements at the heart of NCTM's Process Standards and its philosophy of mathematics education.

I cannot include here every relevant problem with the very idea of national curriculum standards, why they will not address the ills they are supposed to cure, and why we as a profession and nation should be highly skeptical of their deep ties to test makers and publishers who both helped create CCSSM and now stand to reap untold millions from them. But a survey of recent interviews with and articles and columns by Alfie Kohn, Yong Zhao, Anthony Cody, Lynn Stoddard, and Jack Hassard, among others, indicates that many respected progressive educators have serious doubts about where the Common Core promises to take us.

Why so little skepticism from NCTM, NCSM, NCTE, and other major organizations of teachers of mathematics and English? Why are we forced to accept these alleged standards as a fait accompli? Instead, I say, let a thousand curricula bloom, informed by the exponentially growing media and forms of access thereto. A single approach for everyone serves ideologues and corporations, not our children or our democracy.

Michael Paul Goldenberg
mikegold@umich.edu
Director, RationalMath, LLC
Detroit, MI, May 21, 2012

AN APPLICATION OF EXPONENTIAL FUNCTIONS

The photograph (**fig. 1 [Metz]**) shows nested boxes at the UPS store in Las Vegas, Nevada.

A graphing calculator can help make a quick and interesting mathematics investigation. We list the even integers from 8 through 24 (one dimension of each box) in L_1; their cubes in L_2 (the volume of each box); the cost of each box in L_3; and in L_4 the average cost per cubic inch, L_3/L_2. After plotting the points (L_1, L_4), we determine the exponential regression equation, $y = 0.0125665(0.876548)^x$, and graph the equation. The points are remarkably close to the graph of the regression equation.

James Metz
metz@hawaii.edu
Kapi'olani Community College
Honolulu, HI, Sept. 30, 2011

Fig. 1 (Metz)

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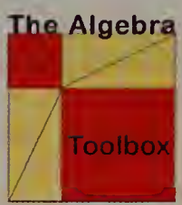

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MORE ON INTRO STATS

In his excellent article "Teaching the Unthinkable in Introductory Statistics" (*MT* October 2011, "Delving Deeper," vol. 105, no. 3, pp. 230–34), William Gratzer provides an accessible overview of the bootstrap, a general-purpose tool for resampling-based inference. This is a key concept of modern statistics that all high school mathematics teachers should be familiar with. I heartily concur with his recommendation that this should be a part of every introductory statistics curriculum.

I take issue, however, with Gratzer's statement that "the bootstrap method does not require large samples," and I have some concerns about the subtleties regarding bootstrap estimation of quantiles (such as the median). As Hesterberg and colleagues state, "We can't trust a bootstrap distribution from a very small sample to closely mimic the shape and spread of the sampling distribution" and "Unless you have expert advice or undertake further study, avoid bootstrapping the median and quartiles unless your sample is rather large" (2008, http://bcs.whfreeman.com/ips6e/content/cat_040/pdf/ips6e_chapter16.pdf).

This form of resampling-based inference is likely to grow in the future and supplement classical large-sample theory. A great deal of research is underway to better understand the bootstrap as well as to extend it to address shortcomings and limitations. Teachers and students should be aware that the bootstrap is a powerful tool that is attractive in many settings but that judgment is required in determining whether the assumptions it makes are appropriate in a given setting.

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ADDITIONAL APPEAL

I enjoyed reading the excellent article "An Ap'peel'ing Activity," by Joshua A. Urich and Elizabeth A. Sasse (*MT* October 2011, vol. 105, no. 3, pp. 189–93), which presented a visual, hands-on approach using oranges to reinforce the classic formulas for the area of a circle and the volume of a sphere. Such activities leave strong impressions on students and can inspire them to look for other mathematical expressions that help describe physical objects.

Fig. 1 (Johnson)

The activity brought to mind another technique I have used over the years to derive the volume of a sphere. It involves the use of a racquetball can (see **fig. 1 [Johnson]**) that contains two racquetballs, each having the same radius r. Assuming that the two balls are tangent to the top, bottom, and sides of the can as well as to each other, we can find the volume of the can as $(4r)r^2\pi$. Racquetballs are typically hollow, so I have on hand a third ball, which I have punched a hole in and filled with water. I pour the water from the ball into the can and note that it mostly fills in all the space between the balls and the can. So if V is the volume of each ball, then we have $3V = 4r^3\pi$, leading to the correct formula for the volume of a sphere.

As a practical matter, I always point out that the water won't completely fill the can because the thickness of the ball prevents the volume of water from being precisely equal to that of the sphere.

Buy enough cans of racquetballs, and this activity could also make a good exercise that might lead students to discover this formula on their own.

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VISUALIZING A TRILLION

Lawrence Lesser's reflection "Visualizing Large Numbers" (*MT* December 2011/January 2012, vol. 105, no. 5, p. 327) triggers a few rather interesting thoughts concerning a large number—one trillion—a number that, sadly, is becoming almost trivial with regard to the increasing national debt.

Here are just two trillion-centered thoughts that may be of some interest:

- One trillion dollars is 1,000 billion dollars, a sum that staggers the imagination. If you could spend one million dollars every day since the beginning of the Common Era (about 734,000 days), you would spend significantly less than one trillion dollars. One trillion is 1,000,000 millions, so you would still have about 266,000 million dollars left, which would last you about another 730 years.
- No human being has ever been able to count to a trillion for the simple reason that if you could utter one number every second, it would take you 32,000 years to count to one trillion. To spell it out: There are 3,600 seconds in 1 hour; 86,400 seconds in 1 day; 31,536,000 seconds in 1 year; 31,536,000,000 seconds in 1,000 years; and 1,009,152,000,000 seconds in 32,000 years.

Fr. Larry N. Lorenzoni, S.D.B.
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San Francisco, CA, Dec. 2, 2011

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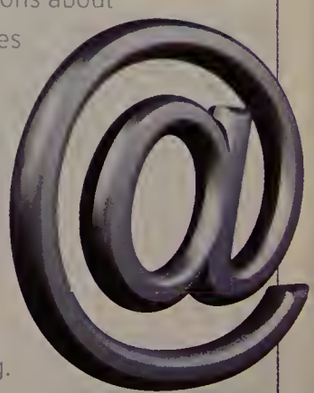

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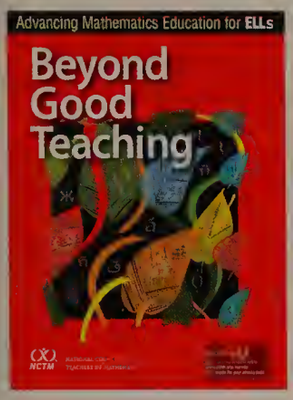

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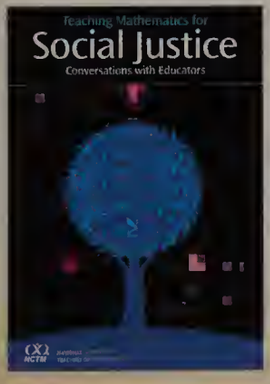

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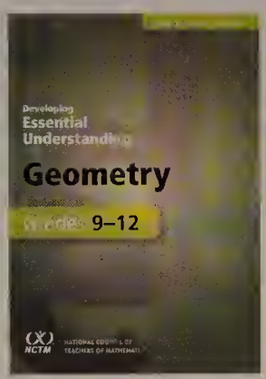

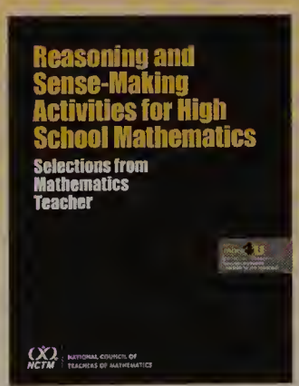

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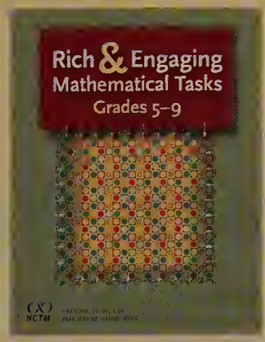

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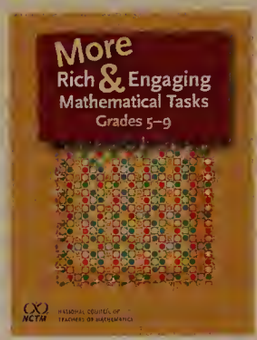

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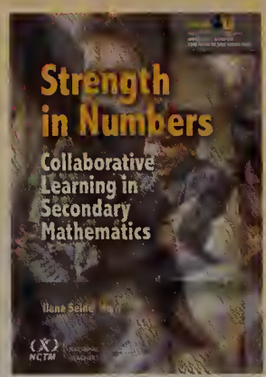

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SOUND off!

Robin S. Kalder

Are We Contributing to Our Students' Mistakes?

Teachers who have been in the classroom for a while know that year after year we see students make the same mistakes that previous students made. To prevent these errors, we try a variety of techniques, but it seems that no matter what we do, they reappear in students' work time and time again. Are we, despite our best efforts, contributing to these mistakes? I believe that we do, indeed, use some practices that inadvertently undermine our best intentions.

MNEMONIC DEVICES AND GENERALIZATIONS

Mnemonic devices and generalizations are used frequently in instruction. They are meant to help students but often hinder them.

One common device is PEMDAS, a mnemonic meant to help students with order of operations. The letters represent the terms *parentheses*, *exponents*, *multiplication*, *division*, *addition*, and *subtraction*. When teachers use PEMDAS, they remind students that multiplication and division have equal importance, but the fact that M comes before D in PEMDAS speaks much louder to students, who often apply multiplication before division instead of applying these operations sequentially from left to right. Of course, performing division before addition, as the spelling of PEMDAS indicates, is correct. How are students supposed to keep the two contradictory interpretations straight when this mnemonic is not connected to a conceptual understanding of order of operations?

Another widely used mnemonic is FOIL, intended to help students multiply binomials procedurally. In this mnemonic, the letters refer to the *first*, *outer*, *inner*, and *last* terms.

Clearly, FOIL does not provide students the conceptual understanding of the distributive property, as evidenced by the fact that many students who learn to multiply binomials this way are not able to find the product of other polynomials. When faced with a problem and directions such as "Multiply the given binomial and trinomial: $(x + 4) \cdot (x^2 + 2x + 3)$," students realize that they cannot use FOIL; they then often claim that they have no method with which to carry out the multiplication. Again, without the conceptual understanding that multiplication of binomials is simply an application of the distributive property, students are not able to extend the concept to multiplying trinomials. In this instance, the students have been foiled.

Generalizations are also frequently used to give students a way to solve problems. For example, the word *of* means to multiply. Although this generalization applies to several different types of word problems, it does not carry over to algebra. When students are introduced to function notation in algebra, one of the fundamental symbols is $f(x)$, read "f of x." Many students view $f(x)$ as multiplication because $2(3)$ does mean 2 times 3 and students have been told that *of* means multiply. These two ideas work together to give students the incorrect belief that $f(x)$ must certainly mean f times x.

IMPRECISE DIRECTIONS

A common complaint among teachers is that students do not read directions. Might one reason be our own negligence? As a mathematics department chair and supervisor of student teachers, I have observed many experienced teachers and preservice teachers in the classroom. Many times I have seen teachers write on an overhead projector or on the front board problems that students are expected to answer, such as this one:

Do now:
1) $3x + 4 = 25$ 2) $4y - 5 = 17$

Students see these problems and begin to work on them. I see these problems, notice that no directions are given, and wonder what the students are doing. Without exception, when I look at these students' work, I see that they are solving for a value for the variable that makes the equation true. Even if the

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teacher had written a direction, it probably would have been the common direction for problems like these: "Solve."

My concern is that such predictable directions minimize the significance of directions and thus reinforce the notion that reading directions is not important. When we give students a problem such as this to solve, we should ask them for the unexpected. A better statement of the problem would be this:

Given $3x + 4 = 25$, solve for $x + 1$.

If we want students to believe that reading instructions is important, we need to make the instructions meaningful, unpredictable, and necessary.

Teachers also minimize the importance of directions when they write and use prevalent "buzz" words. One of the most commonly used directions is "Simplify." How meaningful is this word when used in this way: "Simplify: $\sqrt{12}$"? What we generally want students to do is to rewrite $\sqrt{12}$ as $2\sqrt{3}$. When students are asked which is simpler, $\sqrt{12}$ or $2\sqrt{3}$, they will generally choose $\sqrt{12}$. What they regard as the simplified version of $\sqrt{12}$ is the decimal approximation 3.464.

If we do not want the decimal answer, a more meaningful direction would be, "Rewrite $\sqrt{12}$ using the agreed-on rules for square roots." Of course, this direction would have been preceded by a discussion of the reasons behind the agreed-on rules for working with square roots. If I want students to supply the decimal value as written above, I direct them to find the decimal approximation of $\sqrt{12}$ to the nearest thousandth.

IMPRECISE LANGUAGE

Now think about young students, and especially English language learners, who are told to reduce a fraction. These students have learned that the English word *reduce* means "to make smaller." In reference to fractions, however, these students must understand that reducing a fraction does not make it smaller. And we wonder why students struggle with fractions.

Another example of teachers' ineffective use of language is using nonmathematical language to describe mathematical operations or concepts. When students are asked to evaluate an algebraic expression such as $3x - 4$ when $x = 2$, teachers will direct them to "plug" 2 in for the variable. I looked for the definition of the word *plug* in a mathematics dictionary; it wasn't listed. I did find it in an English language dictionary, but none of the nineteen meanings listed described the mathematical process that students are expected to perform when given this direction. I always direct students to "substitute" 2 for the variable. My payoff comes later, when I present the idea of solving systems of equations by the method of substitution and my students proceed through it without hesitation.

Along this same vein is the word *cancel*. Teachers will say that when finding the difference between polynomials such as $(x^2 + 9)$ and $(x^2 + 2)$, the x^2s will "cancel," so the difference will be 7. The implication is that when two terms "cancel," we are left with nothing. Then when students rewrite a rational expression such as

$$\frac{x^2}{x^2(x+1)}$$

in a simpler form, because it contains factors that do cancel, students will write $(x + 1)$ for their answer. And because the x^2s were canceled, students internalize this to mean that "nothing is left." Therefore, the numerator disappears in the final answer.

Another common use of imprecise language occurs in teaching trigonometry—the mistake that students make when simplifying trigonometric expressions. We ask students to rewrite an expression such as $\sin x/\cos x$, and they respond that this simplifies to tan because the xs cancel. This common mistake can be a consequence of statements that teachers make, such as "Sine divided by cosine equals tangent" rather than "$\sin x$ divided by $\cos x$ equals $\tan x$." This lazy language undermines the interdependence of the function tangent and the angle whose tangent is being taken. When I see students doing this, I tell them that they stopped too soon and that, in addition to cancelling the xs, they should have canceled the ss to get the result "in/co." Students usually get the message.

CONCLUSION

Obviously, there are times when using more casual language in instruction will not create misunderstanding. For example, when we are teaching higher-level mathematics, we certainly can say "minus five" when we really mean "negative five" or, when stating decimal numbers, we can say "two point four" when referring to 2.4. But we can do so safely only after students have completely internalized these basic concepts.

When students are first learning how to work with decimal numbers or signed numbers, however, the use of casual language will undermine our instruction. I will never forget watching a seventh-grade mathematics teacher giving instruction on adding, subtracting, and rounding decimals. During the first half of the lesson, the teacher referred to the decimals in the examples as "two point four plus three point nine," (2.4 + 3.9) or "seven point three two minus five point one six" (7.32 – 5.16). During the second half of the lesson, the teacher assigned rounding exercises and did not understand why students did not know how to locate the tenths place or the hundredths place. Using correct language when stating decimal numbers earlier in the lesson (and, most likely, in previous lessons) could have helped students understand place value.

Clearly, using correct terminology exclusively will not eliminate students' mistakes completely, but it certainly can make an important difference. When students hear correct language being used consistently and are asked to do so themselves, their understanding of mathematical concepts and procedures will no doubt improve.

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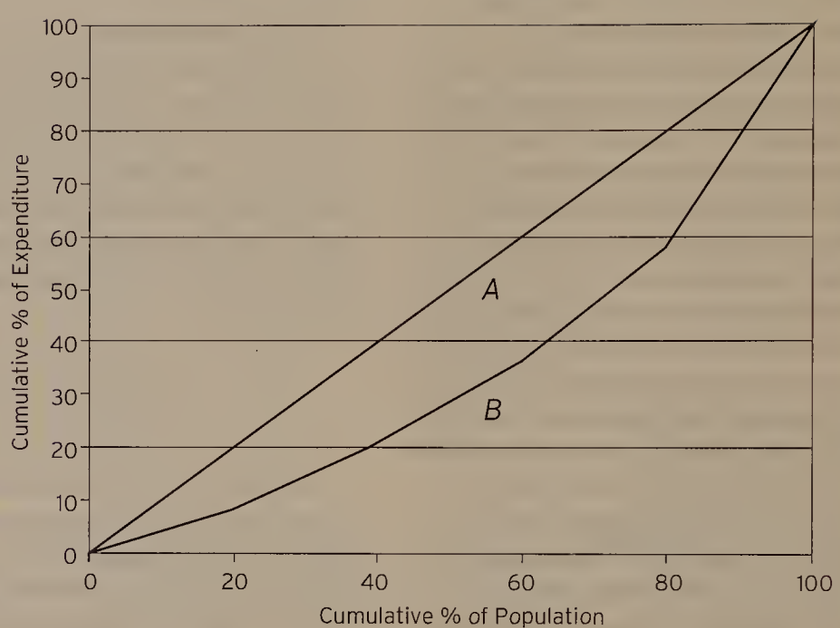
The Editors // Lawrence M. Lesser

Gini Index

The most widely used single measure of inequality is the Gini coefficient. The Gini coefficient is based on the Lorenz curve, a cumulative frequency curve that compares the distribution of a specific variable (e.g., income) with the uniform distribution that represents equality. To construct the Gini coefficient, graph the cumulative percentage of households (from poor to rich) on the horizontal axis and the cumulative percentage of expenditure (or income) on the vertical axis. This gives the Lorenz curve [as shown in the figure]. The diagonal line represents perfect equality. The Gini coefficient is defined as $A(A + B)$, where A and B are as shown on the graph. If $A = 0$, the Gini coefficient becomes 0, which means perfect equality, whereas if $B = 0$ the Gini coefficient becomes 1, which means complete inequality.



Source: info.worldbank.org/etools/docs/library/103072/ch6.pdf

Since the Occupy Wall Street movement began, many news reports have focused on income inequality and the top 1 percent. The Gini coefficient and Lorenz curve provide one way of investigating income inequality.

Gini index (or Gini coefficient) data can be downloaded from a number of sources. Some report the data as a percentage, others as a decimal number between 0 and 1. These websites are revised often and the data updated frequently; we recommend that after accessing the following websites, you then search for "Gini index" within each home page. Some pertinent sources and their websites include the U.S. Census Bureau (www.census.gov); the Central Intelligence Agency (www.cia.gov); and the World Bank (www.worldbank.org).

1. Which two countries in the world have the highest Gini index? Which two have the lowest? What is the Gini index for the United States? What characteristics of these countries do you think might help explain these results?

2. How has the Gini index for the United States changed over time?

3. What is the proper value for the Gini index? Should it increase or decrease?

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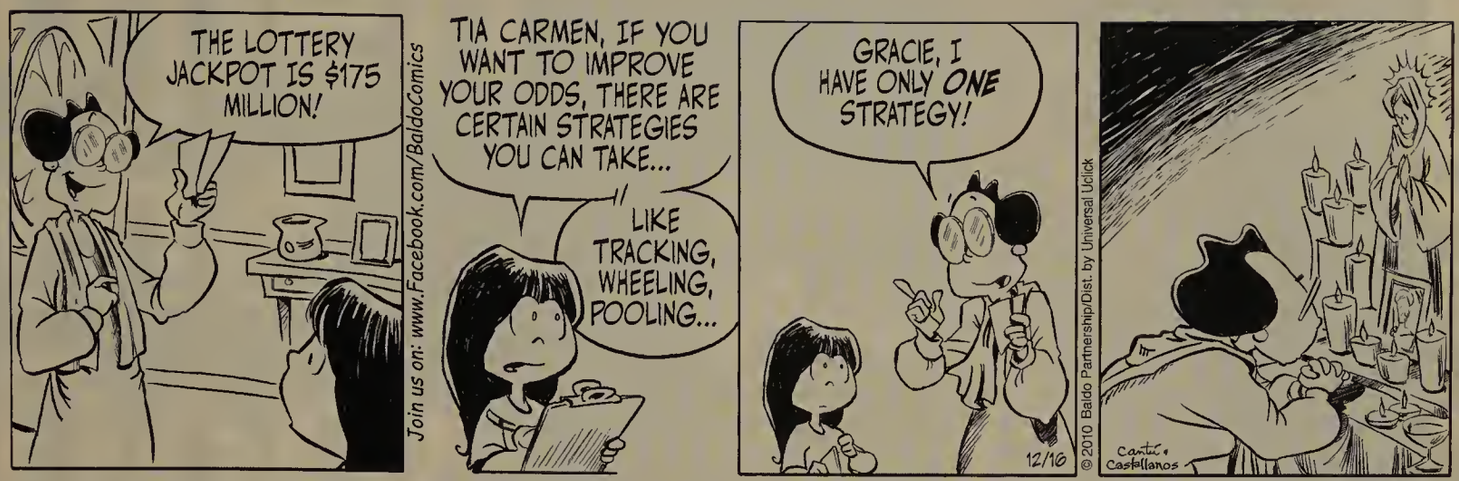
LOTTERY LUNACY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1. *Tracking* means looking up the winning lottery numbers from past drawings (this information is often already tabulated somewhere on the lottery website) to determine what numbers have recently been "hot" or "cold." If the number 5, for example, occurred in the last three drawings, is it more likely than, less likely than, or equally as likely as any other number to occur in the next drawing?

2. What effect will tracking have on your probability of winning? What effect will tracking have on the expected value of your winnings?

3. *Wheeling* is a strategy of betting all the possible combinations of numbers that you somehow believe are "lucky." Suppose that you are playing a lottery in which winning the jackpot requires picking the winning set of 6 numbers from a set of the first 54 natural numbers. How many tickets must you buy to cover all 6-number combinations from a set of 7 lucky numbers? If the lucky numbers are $\{1, 2, 3, 4, 5, 6, 7\}$, write out all the sets of numbers.

4. How many tickets would you have to buy to cover all 6-number combinations from a set of 8 lucky numbers? From a set of 9 lucky numbers?

5. What effect will wheeling have on your probability of winning? What effect will wheeling have on the expected value of your winnings?

6. *Pooling* is a way to combine resources to buy more tickets: If anyone in the group wins, winnings will be divided equally among every person in the group. Participants form a pool of n trustworthy people (at your workplace or a family get-together, for example), and each person buys a ticket, coordinating to make sure that no two tickets have the same set of numbers. What effect will pooling have on your probability of winning?

7. What effect will pooling have on the expected value of your winnings?

8. More lottery information can be found in Lesser's article "Take a Chance by Exploring the Statistics in Lotteries" (*Statistics Teacher Network,* a joint publication of NCTM and the American Statistical Association, no. 65, http://www.amstat.org/education/stn/pdfs/STN65.pdf). What other connections between statistics and the lottery can you make?

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MEDIA CLIPS solutions

"Gini Index" answers

1. Answers may vary. **Table 1 ("Gini Index")** shows one source's most recent data for the distribution of family income for five countries.

   In general, statistics at the extremes are more likely to occur when the sample space is relatively small. In the Seychelles Islands, the owner of a successful tourist hotel might well earn many times more than the neighboring fishermen. Namibia has rich deposits of gem-quality diamonds, yet much of the population depends on subsistence agriculture to survive, and its mortality rate is high as a result of AIDS.

   The populations of both Montenegro and Sweden are fairly homogeneous. Montenegrins voted to secede from a federation with Serbia in 2006. Swedes enjoy a high standard of living by virtue of the many high-tech industries located in Sweden and the extensive welfare benefits available to Swedes. Further, Sweden has managed to avoid participation in any war for almost two centuries.

   The population of the United States is large (the third largest in the world after China and India) and diverse (80% white, 13% black, 4% Asian, and about 15% Hispanic of any race). The American economy has produced a significant number of millionaires and billionaires, yet 15% of the population lives below the poverty line.

Table 1 ("Gini Index")

Nation	Gini	Population
Namibia	70.7	2,147,585
Seychelles	65.8	89,188
United States	45.0	313,232,044
Montenegro	24.3	657,394
Sweden	23.0	9,088,728

Source: CIA's *World Fact Book*

2. Since 1975, nearly all gains in U.S. household income have gone to the top 20% of households. According to the U.S. Census Bureau, the Gini index for income in 1968 was 38.6 and had risen to 47.0 by 2006. (This result can occur because the wealthy are getting wealthier, the poor are falling behind, or both.)

3. Although such questions may indeed be interesting, important, and timely, they are not mathematical questions; the answers are likely to become a matter of political ideology, moral philosophy, or economic theory. Mathematics may, however, help clarify the issues and ground the discussion in reality.

"Lottery Lunacy" answers

1. Equally likely. Each drawing is independent; the balls do not remember their history. "Gambler's fallacy" is a principle that assumes that the law of averages applies to random events. In the lottery context, gambler's fallacy would imply that if the outcome 5 had occurred frequently in recent lottery drawings, it is less likely to occur in the near future. A gambler who believes in the fallacy might provide an answer of "less likely." Someone who believes in streaks may give the incorrect answer of "more likely."

2. Tracking will have no effect on either your probability of winning or the expected value of your winnings.

3. ${}_7C_6$, or 7. The seven 6-number combinations would be {2, 3, 4, 5, 6, 7}, {1, 3, 4, 5, 6, 7}, {1, 2, 4, 5, 6, 7}, {1, 2, 3, 5, 6, 7}, {1, 2, 3, 4, 6, 7}, {1, 2, 3, 4, 5, 7}, and {1, 2, 3, 4, 5, 6}. The number of ways to choose 6 items from 7 is the same as the number of ways to leave out 1 item from 7.

4. Picking 6 numbers from 8 is equivalent to ${}_8C_6 = 28$, and picking 6 numbers from 9 is the same as ${}_9C_6 = 84$.

5. For the example given in problem 3, you would have increased your probability of winning the jackpot by a factor of 7 (as you would by buying any set of 7 nonidentical tickets). However, because you also spent 7 times as much to buy the tickets, the expected value of your winnings remains the same.

6. The total number of tickets bought is n, so your chances of winning a share of the jackpot are now n times greater than if you had bought a single ticket by yourself.

7. If someone in your group wins, your group's winnings will be divided equally among the n people in the group, so you will get $1/n$ of the group's winnings. Although the probability of winning is multiplied by n, you get only $1/n$ of the winnings; thus, the value of your expected winnings is no different from playing the lottery by yourself. Still, some players prefer this approach anyway because they would rather have a bigger chance at a large prize than a smaller chance at a really large prize.

8. Answers will vary.

LAWRENCE M. LESSER, Lesser@utep.edu, who contributed "Lottery Lunacy," is a statistics and mathematics education professor at The University of Texas at El Paso.

Open the Door and Keep It Open

CALL FOR MANUSCRIPTS

Mathematics Teacher is eager to publish articles illustrating ways in which teachers of entry-level courses open the door for students to learn mathematics. These courses are critical to fostering students' pursuit of and love for learning mathematics through the high school years and beyond.

What instructional methods do you find effective when teaching prealgebra, algebra, geometry, or first- and second-year integrated courses? What strategies are successful in addressing the needs of all students in our classrooms?

The *MT* Editorial Panel is soliciting manuscripts that address any of the following topics:

- Successful strategies for setting, communicating, and supporting high expectations for mathematical learning, especially for students who struggle
- Innovative ways to teach concepts in entry-level courses that work with students of all academic levels within the classroom
- Sample lessons or lesson sequences that spur students' mathematical thinking, even after the bell rings
- Approaches that build on students' interests or community heritage
- Examples of noteworthy lessons based on Common Core State Standards that have supported student learning
- Tips for encouraging students' multiple inquiry approaches built on a foundation of reasoning and sense making
- Classroom ideas that inspire students to take mathematics courses beyond minimum graduation requirements
- Use of manipulatives, multimedia, or technology that make mathematics accessible, hands-on, or interactive for students

Share your best lessons or strategies so that we can open the door and keep it open for *all* students to learn mathematics.

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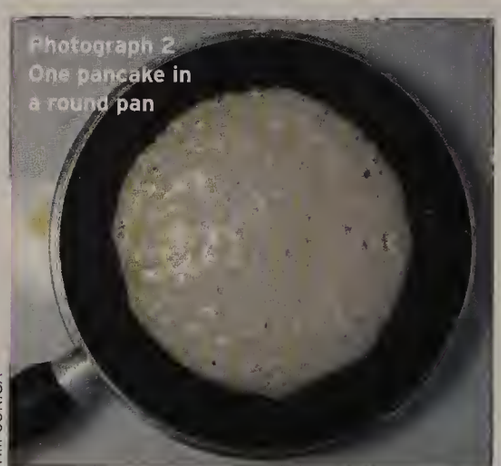
MATHEMATICS teacher

Tim Corica

Making Hay While the Pancakes Cook

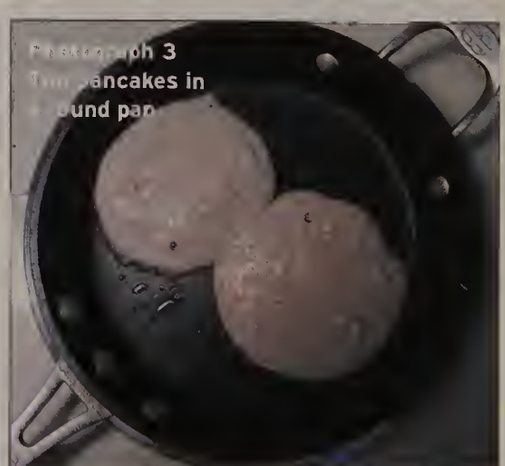
HARRELL RENTZ

Photograph 1
Hay shed, rural North Carolina



TIM CORICA

Photograph 2
One pancake in a round pan



TIM CORICA

Photograph 3
Two pancakes in a round pan

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

Edited by ***Ron Lancaster,*** *ron2718@nas.net, University of Toronto, ON, Canada, and* ***Brigitte Bentele,*** *brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY*

Photograph 1 shows a hay shed in rural North Carolina. The shed portion is a section of a cylinder, and the hay bales are constructed in the form of smaller cylinders. From the front, the roof of the shed forms an arc of a circle, and the bales form smaller circles; see the diagram (**fig. 1**) from the website of SteelMaster, the shed manufacturer. Diagrams like this help farmers arrange bales in a particular shed to maximize the number that can be stored.

1. Assume that the shed roof and the ground beneath it form a semicircle. In **figure 1**, the diameters of the bales and of the semicircle are 5 and 42 feet, respectively. What percentage of

the area of the semicircle is occupied by the circular bales of hay? Note that the diagram shows space between adjacent circles, but the photograph shows none. In answering these questions, assume that the circles are tangent with no space between.

2. In this and the following problems, we assume that the radius of the semicircle is 1 unit. What is the radius of the largest hay bale that will fit in the shed? What is this radius if we had to fit two bales in the shed? Which arrangement (one bale or two) uses up the larger percentage of the area of the semicircle?

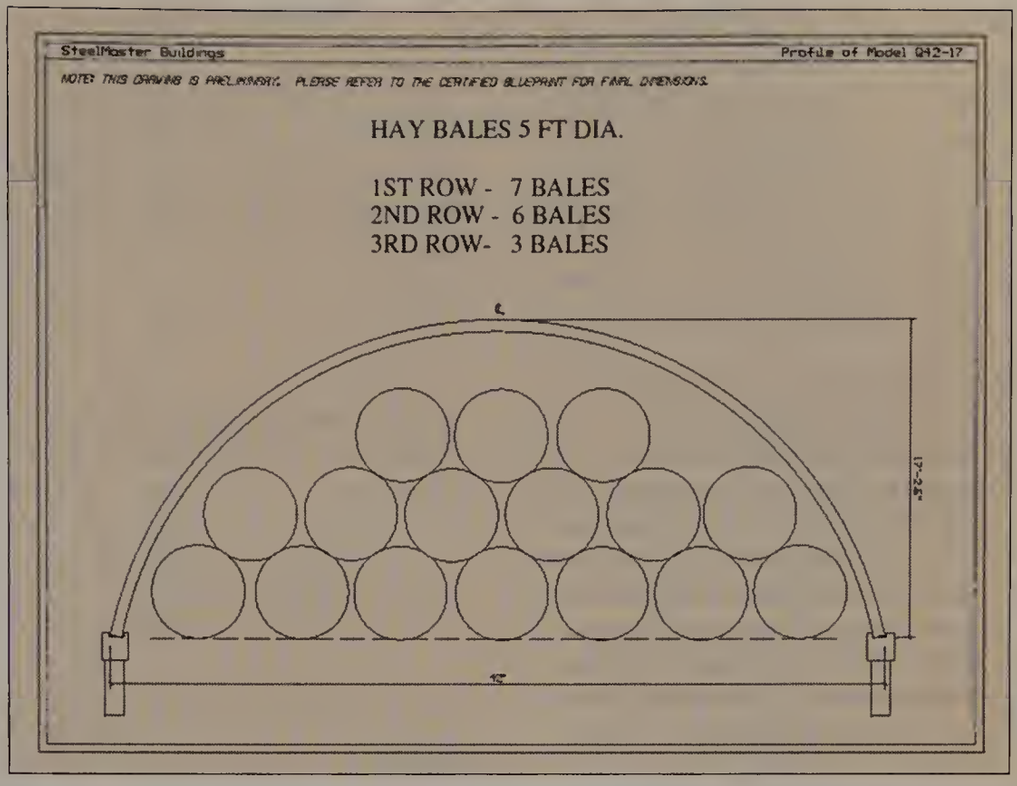
3. Now consider three hay bales. Determine the radius of the largest hay bale such that three bales could be lined up on the ground at the bottom of the shed. Determine the proportion of the area of the semicircle occupied by these three bales.

4. If we stack the three hay bales so that the middle bale is lifted off the ground as much as possible, we can fit three even larger bales into the shed. Find the diameter of these bales and the proportion of the area of the semicircle that they occupy.

5. Devise and analyze an efficient packing scheme for four bales in a semicircular shed. Can you find another? Are the solutions equivalent? (Finding the value for the radius may involve trigonometry; dynamic geometry software might be useful too.)

   Similar questions arise when considering making round pancakes in a round pan (see **photographs 2** and **3**). Obviously, we use 100% of the pan's surface if we make a single, large pancake, but this is often too big for one person to eat.

6. Assume that the pan has a radius $R = 1$ unit. Find the maximum radius of each of two pancakes made in this pan (see **photograph 2**) and the percentage of the pan's cooking surface that is occupied by the pancakes.


SteelMaster Buildings
Profile of Model 942-17
NOTE: THIS DRAWING IS PRELIMINARY. PLEASE REFER TO THE CERTIFIED BLUEPRINT FOR FINAL DIMENSIONS.
HAY BALES 5 FT DIA.
1ST ROW - 7 BALES
2ND ROW - 6 BALES
3RD ROW- 3 BALES

STEELMASTER INC.

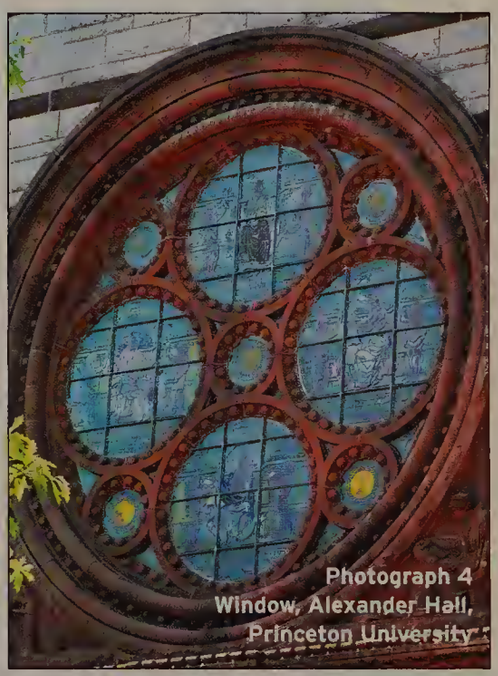
Fig. 1 The hay shed manufacturer provides this diagram for farmers.

7. Solve this problem for three, four, five, and six pancakes. Make a table of your results, showing the pancake radius and the percentage of the pan's cooking surface that the pancakes occupy.

8. An interesting result occurs if we try to arrange seven circular pancakes in a circular pan in the most efficient manner possible. What would this arrangement be? Find the percentage of the area of the pan's cooking surface occupied by seven pancakes, and add this result to your table. What do you notice about the results for seven pancakes?

Extension

Different arrangements can be produced if we relax the restriction that every small circle must have the same radius. **Photograph 4** shows a stained-glass window in Alexander Hall at Princeton University in Princeton, New Jersey. Ignoring the thickness of the brickwork, find the two different sizes of smaller circles and determine the percentage of the window's area that is enclosed by smaller circles. Design your own stained-glass window using only tangent circles.



Photograph 4
Window, Alexander Hall, Princeton University

TIM CORICA

Exploration

Packing problems have a long history in mathematics, and many problems remain unsolved. In the case of circles in a circle, the perfect solution for nineteen smaller circles was proved in 1999, whereas the perfect solution for twenty circles is still unknown. The website http://packomania.com contains illustrations and tables of solutions, many of which have been found by computer but have yet to be proven the best possible.

MATHEMATICAL LENS solutions

1. We can fit sixteen bales into the semicircle, each with area $A = \pi \cdot (2.5)^2 = 25\pi/4 \approx 19.63$ ft.^{2} for a total of about 314 ft.^{2}. The semicircular shed has area $A = \pi \cdot (21)^2/2 = 441\pi/2 \approx 692.72$ ft.^{2} So the fraction is $f = 314.16/692.72 \approx 45\%$.

2. If only a single (very large) bale of hay were placed in the shed, it could be arranged as shown in **figure 2**. Clearly, if the shed has radius 1 unit, the diameter of the hay bale is also 1 unit, so the radius is 1/2 unit. If we use R to represent the radius of the semicircle and r for the radius of the full circle representing the hay bale, the percentage of the area of the semicircle occupied by the hay bale is

$$f = \frac{\pi r^2}{\frac{1}{2}\pi R^2} = \frac{\pi \cdot \left(\frac{1}{2}\right)^2}{\frac{1}{2}\pi \cdot 1^2} = \frac{1}{2} = 0.5 = 50\%.$$

If two bales are placed in the shed, they can sit side by side (see **fig. 3**).

The small red triangle is a 45°-45°-90° triangle, so we know the ratio of the sides. When a circle is internally tangent to another circle, their centers and the point of tangency are collinear. (Can you prove this theorem?)

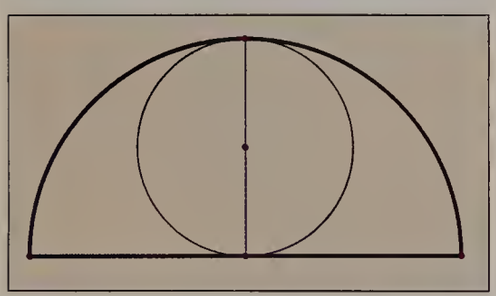


Fig. 2 One large bale would be centered in the shed.

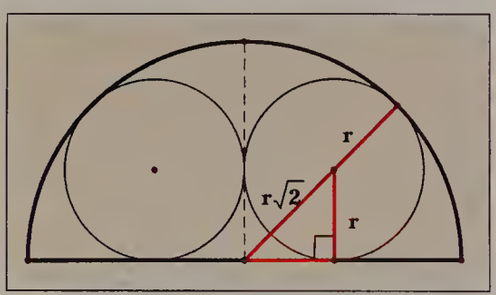


Fig. 3 Two hay bales can be arranged side by side.

The legs of our triangle are each r, and, using the 45°-45°-90° ratios, we find that the hypotenuse is $r\sqrt{2}$. If the radius of the semicircle, R, is 1 unit, then $r + r\sqrt{2} = 1$ and

$$r = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1.$$

The percentage of the area of the semicircle occupied by the hay bales is

$$f = \frac{2 \cdot \pi r^2}{\frac{1}{2}\pi R^2} = \frac{2 \cdot \pi \cdot \left(\sqrt{2}-1\right)^2}{\frac{1}{2}\pi \cdot 1^2}$$
$$= 12 - 8\sqrt{2} \approx 68.63\%.$$

Thus, two smaller hay bales use the space more efficiently than a single large bale.

Figure 4 illustrates another way of arriving at the result. Perpendicular tangent segments are drawn from C, forming two squares, $ABCD$ and $OECF$; OC equals $R\sqrt{2}$ or $\sqrt{2}$ when $R = 1$, and therefore $r = \sqrt{2} - 1$. (Can you prove that $ABCD$ is a square?)

3. In the case in which three circles are lined up along the diameter (see **fig. 5**), we can draw a triangle as we did in the two-circle case. However, we do not know what the angles are, so we cannot use special right-triangle ratios as before. We do know that the longer leg of the triangle is $2r$ because the endpoints of this leg and the two centers form the vertices of a rectangle.

We can use the Pythagorean theorem: $(2r)^2 + r^2 = (R - r)^2$. Letting $R = 1$ and solving, we find that $r = (-1 + \sqrt{5})/4$. This size and layout give us the percentage of the semicircle that the hay bales occupy:

$$f = \frac{3 \cdot \pi r^2}{\frac{1}{2}\pi R^2} = \frac{3 \cdot \pi \cdot \left(\frac{-1+\sqrt{5}}{4}\right)^2}{\frac{1}{2}\pi \cdot 1^2}$$
$$= \frac{9 - 3\sqrt{5}}{4} \approx 57.29\%$$

This packing arrangement is slightly less efficient than the two circles case.

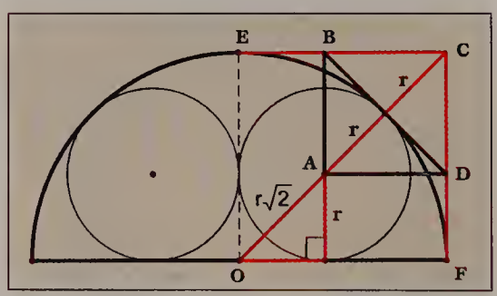


Fig. 4 Recognizing that $ABCD$ and $OECF$ are squares helps students find r.

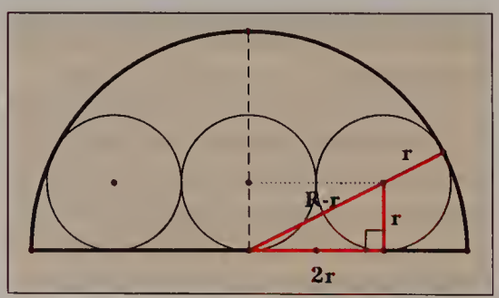


Fig. 5 The Pythagorean theorem provides a way to determine r when we have three circles.

4. See the arrangement in **figure 6**. Here the middle hay bale seems to be propped up on the two lower bales. This arrangement allows us to make the bales slightly bigger. To find out how much bigger, we note that all three bales touch the semicircle, so we can simply look for the largest circle that will fit in a 60° pie-shaped wedge. We again draw a triangle (see **fig. 6**).

Here we make use of the fact that the wedges are 60°, so the smallest angle of the triangle is 30° and the side opposite is equal to r. Using 30°-60°-90° triangle ratios, we have $2r + r = R$; if $R = 1$, we have $r = 1/3$. The new

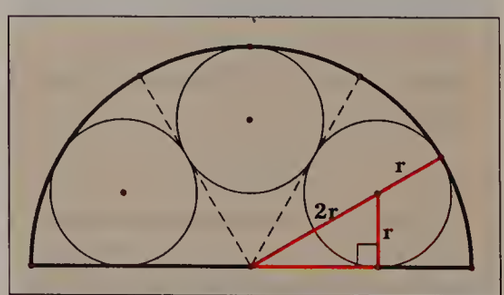


Fig. 6 With these three bales, we return to a special right-triangle situation.

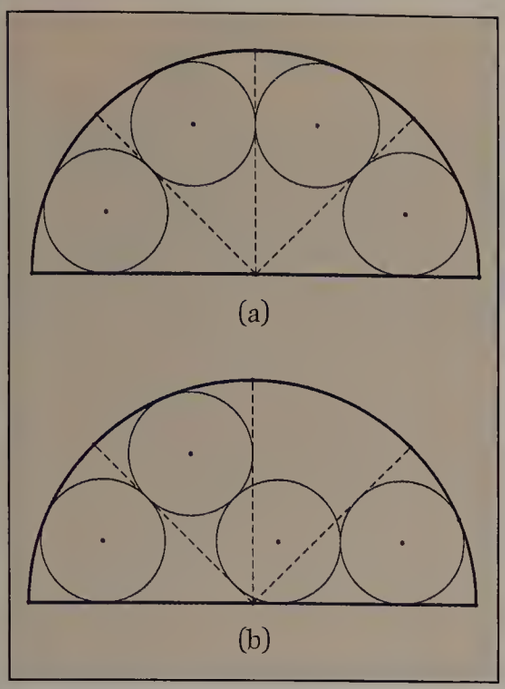


Fig. 7 With four bales, two arrangements are possible: (**a**) and (**b**). The bales will have the same dimensions in either arrangement, and these dimensions can be found using trigonometry.

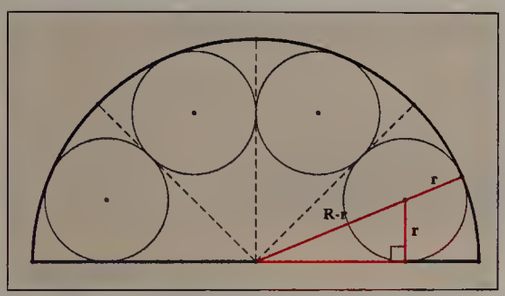


Fig. 8 Right-triangle trigonometry produces a solution when there are four bales.

percentage of the area of the semicircle occupied by the hay bales is

$$f = \frac{3 \bullet \pi r^2}{\frac{1}{2}\pi R^2} = \frac{3 \bullet \pi \bullet \left(\frac{1}{3}\right)^2}{\frac{1}{2}\pi \bullet 1^2} = \frac{2}{3} \approx 66.66\%,$$

a substantial improvement over the three-bales-in-a-row case. It is interesting to compare this with the two bales case. In the three bales case, the circles are slightly smaller, and we would think that they would fit more neatly into the nooks and crannies of the semicircle. However, this is a slightly less efficient packing arrangement than the two circles case (66.66% vs. 68.63% of the area of the semicircle).

5. Two possible solutions are shown in **figure 7**. We leave it to readers to prove that the configuration shown in **figure 7b** produces equivalent results. The central angle in the small red triangle in **figure 8** measures 22.5°. Letting $R = 1$, we obtain $r \approx 0.2768$, leading to about 61.28% of the area of the semicircle occupied by the hay bales.

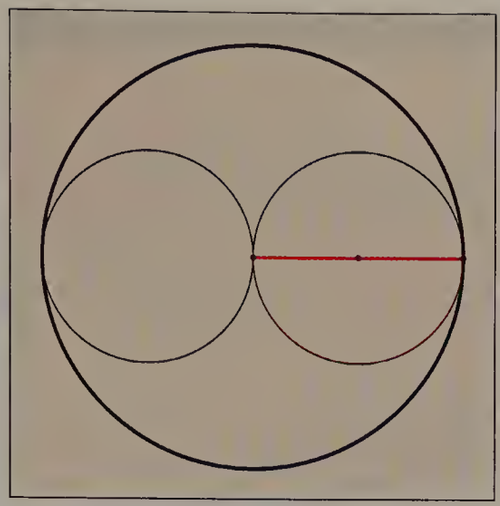


Fig. 9 The radius of the pan is the diameter of each pancake.

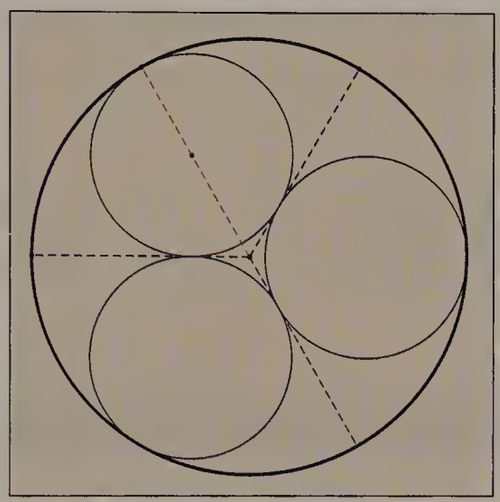


Fig. 10 Special triangles are again involved when we work with three pancakes.

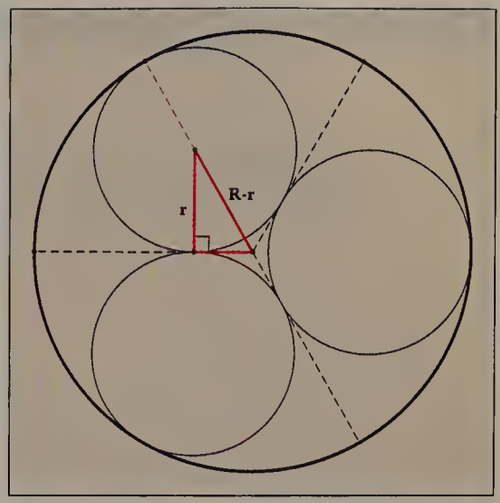


Fig. 11 The triangle is a 30°-60°-90° triangle, so the special ratios can be used.

6. The arrangement of pancakes is shown in **figure 9**. The radius of each pancake is half the radius of the pan. So the percentage of the pan's surface area that the pancakes occupy is

$$f = \frac{2 \bullet \pi r^2}{\pi R^2} = \frac{2 \bullet \pi \bullet \frac{1}{2}^2}{\pi \bullet 1^2} = \frac{1}{2} = 50\%.$$

7. The arrangement for three pancakes is shown in **figure 10**. To find the radius, we construct a triangle (see **fig. 11**). The three circles are rotated 120° from one another, so we know that the larger acute angle is 60°. Using the ratios from the 30°-60°-90° triangle, we can obtain the following proportion:

$$\frac{r}{R-r} = \frac{\sqrt{3}}{2}$$

Using $R = 1$ and solving for r, we find that

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Table 1 Pancake Ratios

N	Radius (R = 1)	Percentage of Surface Area Occupied by Pancakes
2	1/2	$1/2 = 50.00\%$
3	$-3+2\sqrt{3}$	$63-36\sqrt{3} \approx 64.62\%$
4	$\sqrt{2}-1$	$12-8\sqrt{2} \approx 68.63\%$
5	$\frac{\sin(\pi/5)}{1+\sin(\pi/5)}$	$\frac{5\sin^2(\pi/5)}{(1+\sin(\pi/5))^2} \approx 68.52\%$
6	1/3	$2/3 \approx 66.67\%$

$$r = \frac{\sqrt{3}}{2+\sqrt{3}} = 2\sqrt{3}-3.$$

Thus, the percentage of the pan's surface area occupied by the three pancakes is

$$f = \frac{3 \cdot \pi r^2}{\pi R^2} = \frac{3 \cdot \pi \cdot \left(2\sqrt{3}-3\right)^2}{\pi \cdot 1^2}$$
$$= 63-36\sqrt{3} \approx 64.62\%.$$

In fact, for any number of pancakes we can repeat the process of subdividing the circle into congruent sectors of the same number of desired pancakes and inserting circles accordingly. For four pancakes, the triangle will be 45°-45°-90°; for five pancakes, the triangle will be 36°-54°-90°, so we will need a value for sin 36°; and for six pancakes, the triangle is 30°-60°-90°. See **table 1**, where N represents the number of pancakes.

8. In the arrangement for six pancakes around the edges of the pan, the radius of each pancake is 1/3, which means that there is just enough space to put a seventh pancake in the middle. The solution for seven pancakes is a radius of 1/3, for a fraction of $7/9 \approx 77.78\%$, the largest fraction we have seen for any number of pancakes greater than one large pancake. In fact, it is believed that to get a greater percentage, one would need to place nineteen pancakes in the pan, an unrealistic number for most pans!

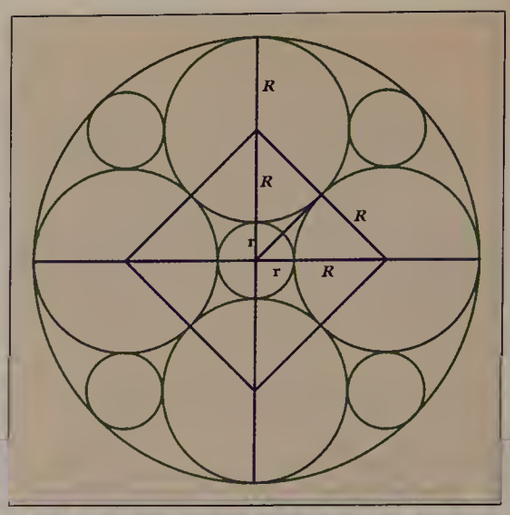


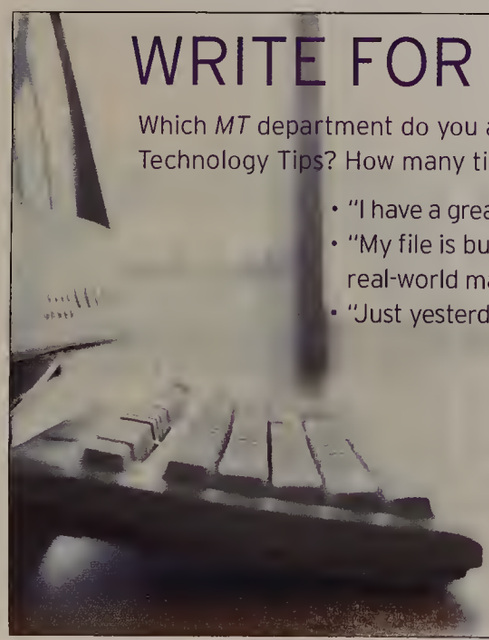
Fig. 12 The stained-glass window can be abstracted in this way.

Extension

As shown in **figure 12**, the radius of the stained-glass window in which the two sizes of smaller circles are contained is $2R + r$, where R is the radius of each of the four circles and r is the radius of each of the five smaller circles. Further, by the 45°-45°-90° triangle ratios, $r + R = R\sqrt{2}$ or $r = R(\sqrt{2} - 1)$. Let the radius of the window be 1 unit. Then $2R + R(\sqrt{2} - 1) = 1$, and therefore $R = \sqrt{2} - 1$ and $r = 3 - 2\sqrt{2}$. The ratio of the areas of the smaller circles to the area of the window is

$$\frac{4(\sqrt{2}-1)^2\pi + 5(3-2\sqrt{2})^2\pi}{\pi} \approx 83.35\%.$$

TIM CORICA, tcorica@peddie.org, chairs the mathematics department at Peddie School in Hightstown, New Jersey. He teaches mathematics and computer science and is interested in connecting mathematics to science applications.



WRITE FOR A DEPARTMENT

Which *MT* department do you always read first? Calendar? Media Clips? Technology Tips? How many times have you thought–

- "I have a great problem for the Calendar,"
- "My file is bulging with newspaper clippings for bringing real-world mathematics into the classroom," or
- "Just yesterday, I thought of a new calculator approach."

Share your experience and expertise with colleagues: Write for a department. If you would like more information on how to get started, go to nctm.org/publications/content.aspx?id=10440. If you have an idea that you want to send in, check the submission instructions that accompany each department in the journal.

more 4U

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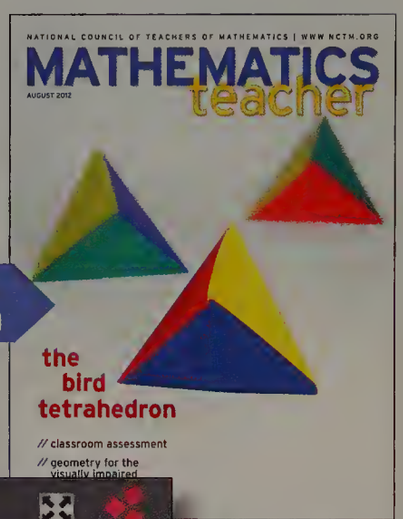

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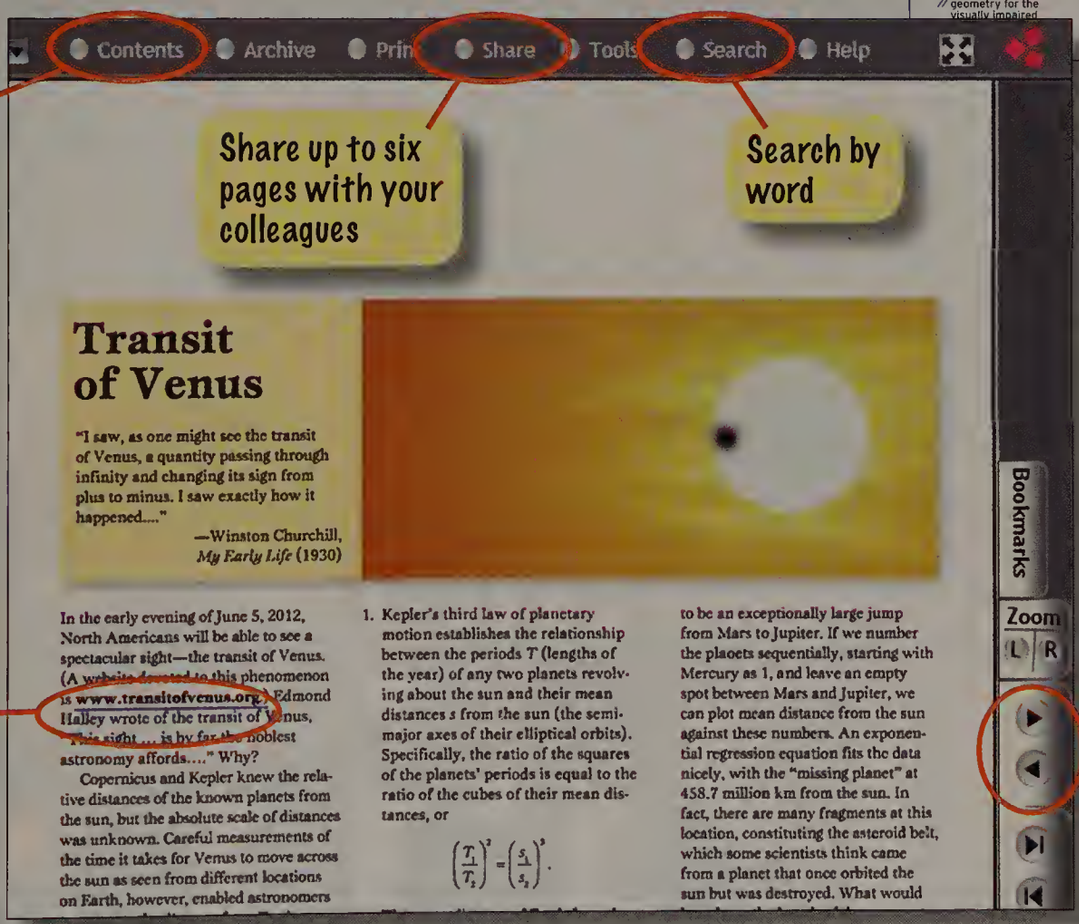
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R
Transit of Venus
"I saw, as one might see the transit of Venus, a quantity passing through infinity and changing its sign from plus to minus. I saw exactly how it happened...."
—Winston Churchill, My Early Life (1930)
In the early evening of June 5, 2012, North Americans will be able to see a spectacular sight—the transit of Venus. (A website devoted to this phenomenon is www.transitofvenus.org.) Edmond Halley wrote of the transit of Venus, "This sight ... is by far the noblest astronomy affords...." Why?
Copernicus and Kepler knew the relative distances of the known planets from the sun, but the absolute scale of distances was unknown. Careful measurements of the time it takes for Venus to move across the sun as seen from different locations on Earth, however, enabled astronomers
1. Kepler's third law of planetary motion establishes the relationship between the periods T (lengths of the year) of any two planets revolving about the sun and their mean distances s from the sun (the semi-major axes of their elliptical orbits). Specifically, the ratio of the squares of the planets' periods is equal to the ratio of the cubes of their mean distances, or
$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{s_1}{s_2}\right)^3.$$
to be an exceptionally large jump from Mars to Jupiter. If we number the planets sequentially, starting with Mercury as 1, and leave an empty spot between Mars and Jupiter, we can plot mean distance from the sun against these numbers. An exponential regression equation fits the data nicely, with the "missing planet" at 458.7 million km from the sun. In fact, there are many fragments at this location, constituting the asteroid belt, which some scientists think came from a planet that once orbited the sun but was destroyed. What would

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Three-Dimensional Printi

A Journey in Visualization

Using simple materials, a Mathematica software application, and their knowledge of function transformations, students design and create real mathematical sculptures.

Adam Poetzel, Joseph Muskin, Anne Munroe, and Craig Russell

Imagine high school students glued to computer screens—not playing video games but applying their mathematical knowledge of functions to the design of three-dimensional sculptures. Imagine these students engaging in rich discourse as they transform functions of their choosing to design unique creations. Now, imagine these students using three-dimensional printers to make their mathematical sculptures come into being as lasting artifacts of what they have accomplished in mathematics class.

At the University of Illinois at Urbana-Champaign, researchers and educators have successfully collaborated to turn this idea into reality. Using Mathematica® software, a data projector, a magnifying glass, a mirror, a liquid polymer, and about twenty dollars' worth of materials from a hardware store, students can apply their mathematical knowledge to designing and creating their own three-dimensional plastic sculptures.

The prospect of designing and printing three-dimensional sculptures spurs students to interact meaningfully with mathematics. The concept of function transformations is a central theme from algebra through calculus, yet it is one that many students fail to internalize. A major mathematical goal of our project is for students to become expert "function tamers" (CCSSI 2010, Functions standard F-BF3). When discussing this idea with students, we talk about the power of being able to manipulate functions. Just as they have seen lion tamers at the circus coercing wild felines to obey their every command,

JOSEPH MUSKIN

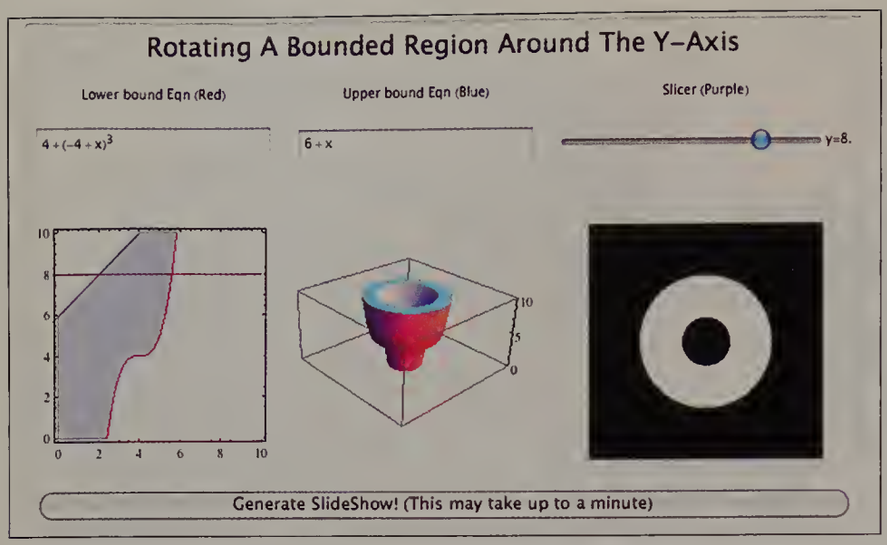

Fig. 1 The student interface window from the Mathematica application shows the bounded region and the solid that results from a rotation about the *y*-axis.

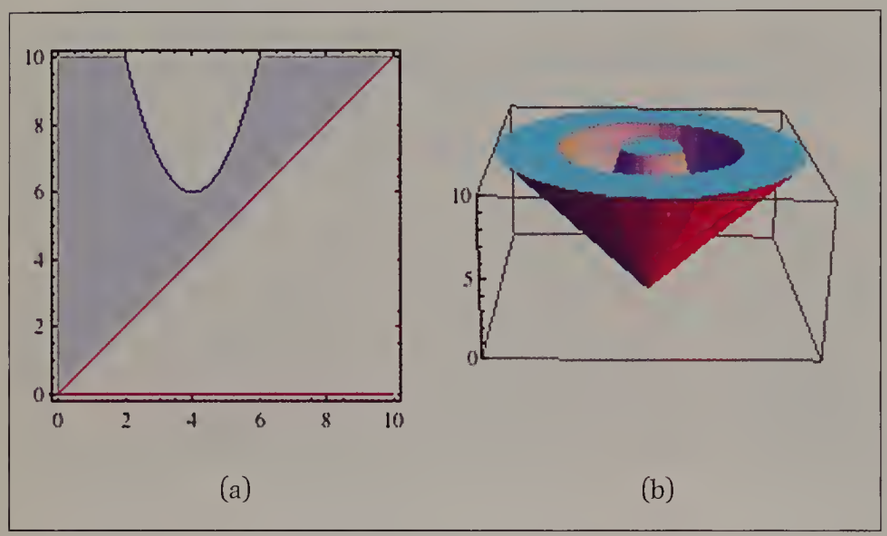

Fig. 2 The shaded area (**a**) creates the solid (**b**) when rotated about the *y*-axis.

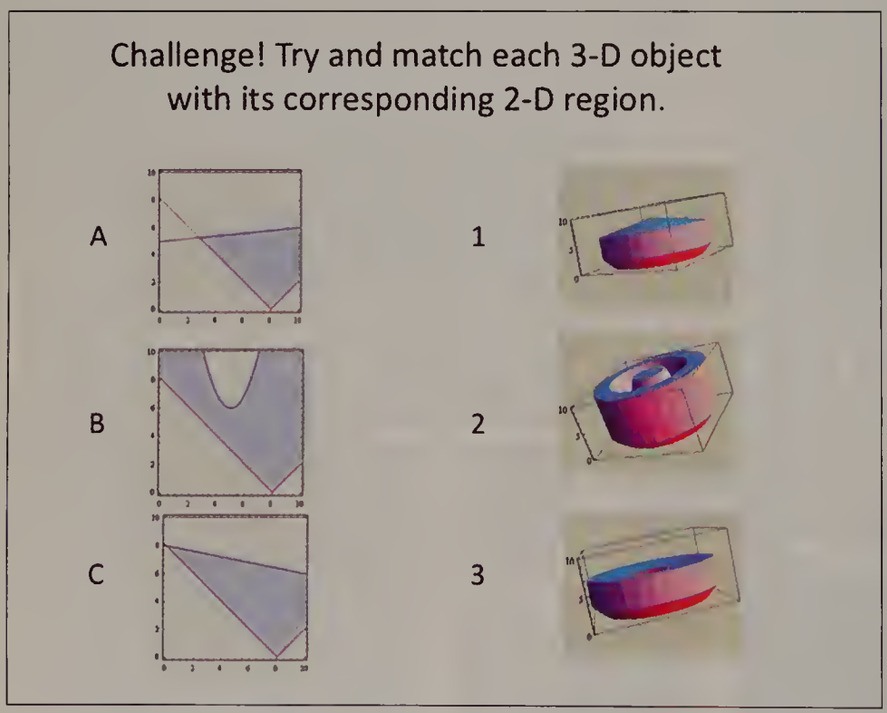

Fig. 3 Students match the two-dimensional region and the resulting solid of revolution. (Answers: A → 3, B → 2, C → 1)

students can become "function tamers," able to transform any function as they please. By learning and practicing the "taming secrets," they will create their own unique mathematical sculptures.

DESIGNING THE OBJECT

As part of this process, students need to be able to turn two-dimensional functions into three-dimensional representations. To accomplish this, we use solids of revolution. A solid of revolution is a three-dimensional figure obtained by rotating a two-dimensional region around an axis. Often, this concept is not discussed until calculus, but with the aid of the right technology, solids of revolution can become accessible to beginning algebra students. The right technology in this instance is a computational program called Mathematica, used in scientific, engineering, and mathematical fields.

A Mathematica application developed by Poetzel and programmers at Wolfram allows users to create a two-dimensional region by defining two functions of their choosing. One function serves as the upper bound, one as the lower bound, and Mathematica shades in the enclosed region (see **fig. 1**). This shaded region is then rotated around the *y*-axis to create a three-dimensional solid of revolution (see **fig. 2**). When using this application, students always work in a window with a predetermined domain and a range from zero to ten. Students can see instant changes in the three-dimensional solid as they transform their functions. By experimenting with polynomial, trigonometric, absolute value, and exponential functions, students can create an endless variety of both simple and complex three-dimensional representations.

These side-by-side two- and three-dimensional representations allow students to make connections between the figures and engage in a powerful visualization experience (CCSSI 2010, Geometry standard G-GMD.4). To help establish these connections, we have developed some classroom discussion prompts in which students match given two-dimensional regions with the corresponding solid of revolution (see **fig. 3**).

PRINTING THE OBJECT

To create a three-dimensional version of an object displayed on the computer screen, the software must also create horizontal cross sections at any given height of the sculpture. Thus, a third representation included in the Mathematica application is one that allows students to control a slider that moves a purple horizontal line in the two-dimensional representation up and down. As students do this, the screen shows a black-and-white representation of the cross section of their three-dimensional sculpture as if it were sliced horizontally

at that height (see **fig. 4**). Once again, the instantaneous updating of all three representations is a powerful exercise in establishing the mathematical connections between the two-dimensional, three-dimensional, and cross-sectional views.

Once students have designed a suitable object, they are ready to begin the printing process. Basic commercial three-dimensional printers can cost tens of thousands of dollars and are highly technical pieces of equipment. The beauty of the inexpensive printer designed by the staff of Nano-CEMMS at the University of Illinois is that it can be made from items that are readily available at most schools; as mentioned earlier, these include a data projector, a magnifying glass, a mirror, a liquid polymer, and about twenty dollars' worth of materials from a hardware store.

The key to this process is that the objects are formed from a light-activated liquid polymer. After students pour the liquid into a beaker, they use a magnifying glass and a mirror to focus and direct the light from a classroom data projector onto the liquid (see **fig. 5**). Wherever the intense light from the data projector shines, the liquid solidifies. The light penetrates only a short distance down into the chemicals, resulting in a thin, solid layer in the exact shape of the light.

Using this liquid and a staging device (see **fig. 5**), students can create a three-dimensional object in the following manner. The Mathematica application creates a slide show consisting of cross sections; the slides begin at the bottom of the three-dimensional object and systematically work up to the last cross section at the top. The platform is placed just under the surface of the liquid. The projected cross section of light forms a pattern, and the liquid solidifies in that pattern.

The stage is then lowered, and the first cross section, which is adhered to the stage, is thus lowered with it. The liquid polymer flows over the top. The process is repeated, and the second layer solidifies and adheres to the first cross section beneath it. This process is repeated to create the entire object, layer by layer. Because the light does not penetrate the liquid too deeply, later layers do not bleed down and solidify open areas from lower layers. This process allows cantilevered structures to be created. In about twenty minutes, students have created a lasting artifact as a reminder of what they have accomplished using mathematics and science (see **fig. 6**). The finished sculptures range from 1 to 3 cm in width and height.

A detailed description of the setup, chemicals used, and plans to make a simple staging device can be found online (http://nano-cemms.illinois.edu/3dprinting) or in Muskin, Ragusa, and Gelsthorpe (2010).

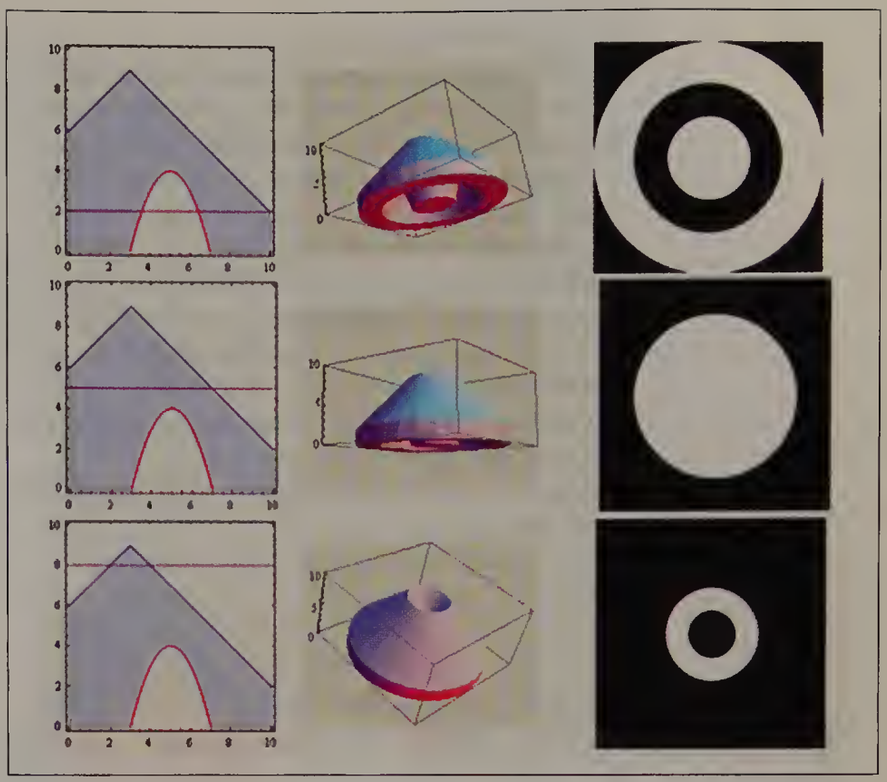

Fig. 4 The purple line in the illustration on the left in each group indicates the height of the cross section to be taken on the solid of revolution pictured in the center illustration. The illustration on the right is the cross section.

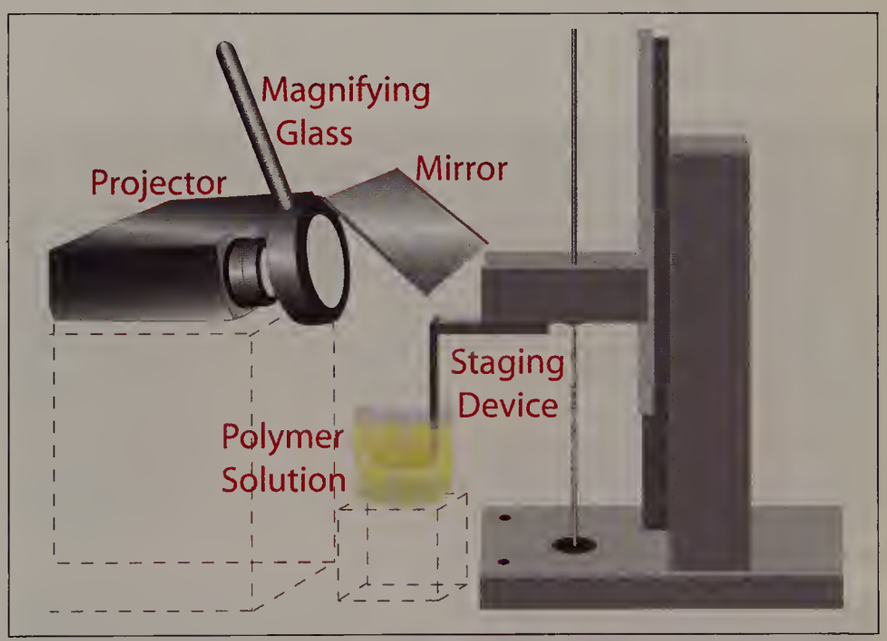

Fig. 5 The equipment needed to create the solid objects is relatively easy to set up.

BRINGING IT TO THE CLASSROOM

We have used this lesson with students ranging from eighth-grade algebra students to advanced calculus students, and here we share our experience working with a non-AP calculus class. We were invited to teach a three-day module preceding the class's unit on solids of revolution. Students in the class were primarily seniors who had successfully completed algebra 2 and trigonometry.

Our goal on day 1 was to pique students' curiosity by introducing them to the three-dimensional

process and letting them know the goal of the mini-unit: to design and construct their own mathematical sculptures. We immediately captured students' attention, and they were eager to get started.

Pairs of students were given laptops and a worksheet entitled "Match-Me" with representations of various two-dimensional regions and three-dimensional solids of revolution (similar to **fig. 3**). Students began by making conjectures about which two-dimensional region on the worksheet corresponded with which three-dimensional solid of revolution. Then, to become expert function tamers, they worked together to write functions on their computer to match the two- and three-dimensional representations on the worksheet and determine whether their conjectures were correct. Students spent a large portion of days 1 and 2 on this task.

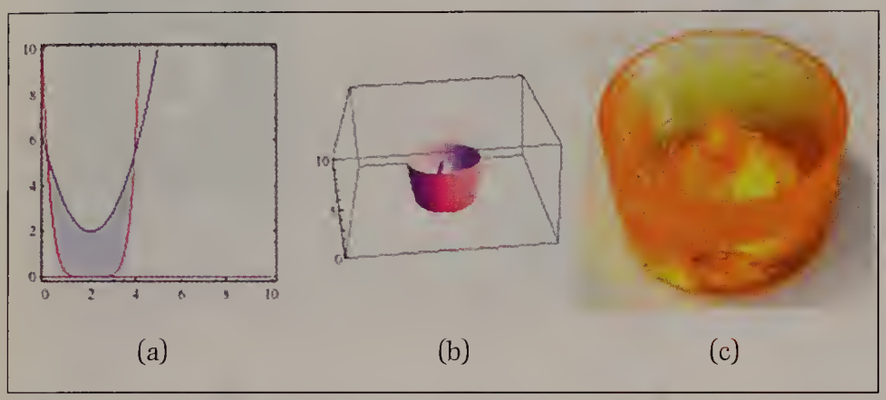

Fig. 6 The process consists of only three steps from start to finish.

Fig. 7 All these objects were created by high school calculus students.

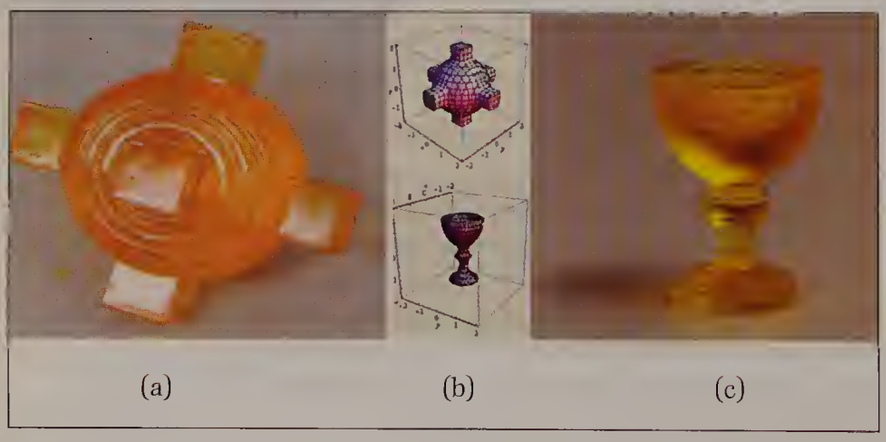

Fig. 8 Students used multivariable inequalities and Boolean operators to make these sculptures.

As expected, students exhibited a wide range of prior knowledge when applying transformations of functions. What they did not know, they were able to learn very quickly because of the real-time feedback from the computer. Student discourse was rich; pairs of students debated which type of functions to use, how to "stretch" or "shrink" their graphs, and which numbers should be positive or negative to move their functions in the desired directions. With each change in their functions, students could immediately view the resulting solid of revolution and better understand the relationship between the two- and three-dimensional representations.

Once a pair of students had successfully "matched" the required representations on the worksheet, they used their enhanced knowledge of transformations to design their own sculptures. Students were mesmerized as they worked to create their three-dimensional objects. The technology allowed this activity to be self-differentiating. Students who had greater mastery of algebraic concepts, instead of finishing sooner, were motivated to create elaborate sculptures by exploring more complex mathematical representations, including transformations of trigonometric and higher-degree functions. It was invigorating to listen to students engage in mathematical discussions and see their excitement when they succeeded in manipulating functions to create interesting objects.

Day 3 of the unit was spent teaching students the three-dimensional printing process and letting them use the printers to create their own unique objects (see **fig. 7**). Each pair of students worked with its own laptop and printing station. Students appreciated having a tangible application of the mathematics they had learned; others enjoyed seeing what could be accomplished by combining mathematics and a little chemistry. The teacher also reported that, after completing this project, students seemed to have a much easier time visualizing the solids of revolution formed in calculus problems. Consequently, students found it much easier to correctly set up and evaluate the integral necessary for calculating the volume of those solids of revolution.

OTHER APPLICATIONS

A second Mathematica application was developed for students studying multivariable functions who were already adept at transformations of functions in the xy-plane. Instead of using functions to create a three-dimensional object by rotating a two-dimensional area around the y-axis, this application allows students to visualize surfaces defined by functions in three-dimensional space. Through the use of multiple inequalities and Boolean operators (And, Or, Not),

Fig. 9 One student used a drawing program to create the cross sections for the Jefferson Memorial.

students can sculpt an object by defining planes, spheres, cylinders, and curved surfaces. Students have used this application to sculpt chalices, rings, and other interesting geometric designs (see **fig. 8**). The power of the real-time feedback given by the software provides a rich learning experience for students; they can instantly view the ramifications of the numbers and symbols they choose for their inequalities. We have used the application with BC calculus students (after they have taken their AP test), calculus 2 and 3 classes, and preservice teachers.

As an exercise in visualization, students can create objects without the use of Mathematica software. Using PowerPoint®, students can design their own black-and-white cross sections one at a time to build three-dimensional structures. Students have created slide shows of cross sections that build miniature replicas of objects such as pirate ships, pianos, chessboards, and the Jefferson Memorial (see **fig. 9**). The Mathematica files, introductory PowerPoint, student worksheets, video tutorials of the entire printing process, and other lesson details are available as free downloads through the University of Illinois (https://nano-cemms.illinois.edu/materials/3d_printing_with_mathematica_full).

CONCLUSION

The most rewarding aspect of this whole process has been the opportunity to observe entire classes of students engaged in a rich mathematical task. NCTM's five Process Standards—Problem Solving, Reasoning and Proof, Communication, Connections, and Representation (NCTM 2000)—come to life as pairs of students use technology to make connections between multiple representations of a mathematical concept. Ideas flow as students work together to analyze how their choice of functions affects their resulting unique sculptures. As students decide on the specific sculptures they wish to create, they engage in meaningful problem solving to determine which functions or inequalities will turn their mental images into real objects. And, for the grand finale, students, using a beaker of chemicals and a data projector to build their objects one layer at a time, are able to see the connections between mathematics and science.

This activity represents an exciting journey in our professional lives. The collaboration between researchers, programmers, teacher-educators, and in-service teachers that turned this idea into reality has been extremely rewarding. Each contributor added a key piece of the puzzle that brought this idea into the classroom and become a memorable landmark in many students' educational path.

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CORE CONVERSA

Jeffrey J. Wanko,
Michael Todd Edwards,
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The Measure-Trace-Algebratize (MTA) approach allows students to uncover algebraic relationships within familiar geometric objects.

As teachers, we all live for "aha!" moments—instances in classrooms when students attain new levels of mathematical understanding and insight. Even more rare and elusive are "gasp!" moments—instructional events so utterly unexpected and surprising that they take one's breath away, compelling students and their teachers to reexamine deeply held beliefs about mathematics content, teaching, and learning.

The Sliding along a Side task was presented at an NCTM conference session for high school teachers:

> Consider square *ABCD* and point *E* on side *CD*. Describe relationships that exist among the various shapes within the figure as *E* moves along segment *CD* (see **fig. 1**). As you slide point *E* along *CD*, what happens to the various measures in the figure? Consider segment lengths, angles, areas, and any other measures you find interesting. (A dynamic sketch of this problem is available at http://tinyurl.com/fig001.)

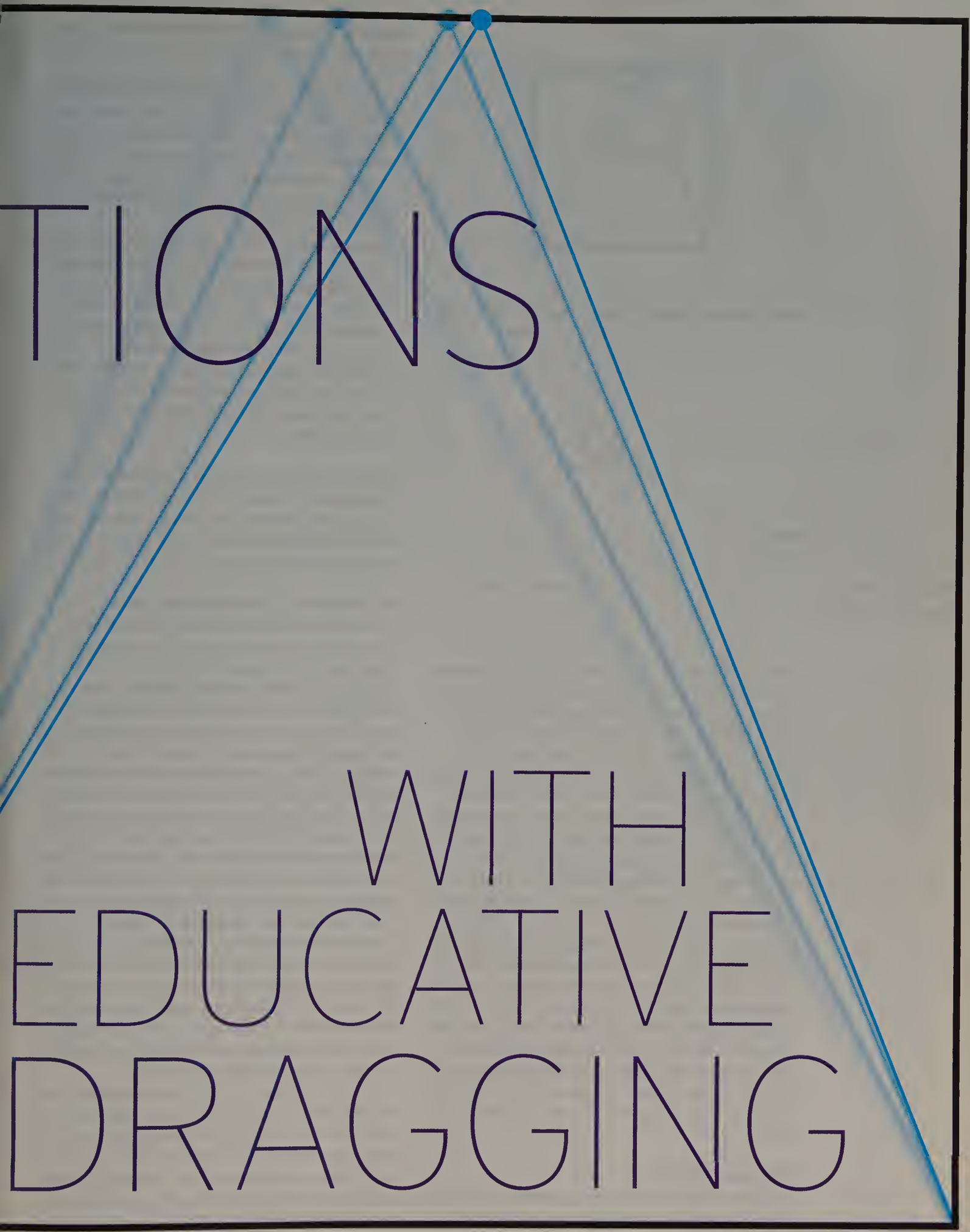
Specifically, we constructed a square *ABCD* and a moveable point *E* on side *CD* of the square. Segments *EA* and *EB* were drawn. For computational ease, we let the side length of the square equal 3 units.

While exploring this task with this group of experienced mathematics educators and classroom teachers, we shared a "gasp!" moment. With the aid of the dynamic mathematics software GeoGebra, participants investigated, in a whole-group setting, various conjectures associated with the task. Our group's collective conversation inspired us to explore new approaches and methods for analyzing the seemingly simple geometric figure. As "obvious" results were proved false, the task generated comments of surprise from audience members and presenters alike.

We used the Measure-Trace-Algebratize (MTA) framework to create the Sliding along a Side task, a framework that other teachers can use to generate rich mathematical contexts for their students. Building on an understanding of the MTA approach, we discuss this task in detail, illustrating the usefulness of such tasks as vehicles for uncovering algebraic relationships within familiar geometric objects (Driscoll 2007; Harper and Edwards 2010; Edwards, Phelps, and Wanko 2010).

EDUCATIVE DRAGGING

Using dynamic mathematics software (DMS), students generate numerous instances of geometric objects by clicking and dragging. In such environments, a square is no longer a single static image on a page. Rather, a dynamic square may be thought of as a collection of quadrilaterals of different sizes and dimensions, each retaining essential qualities of "squareness"—four congruent interior angles and four congruent sides. Although a dynamic square is resized as students drag its vertices, the attributes that make the quadrilateral a square remain unchanged. DMS users learn to look for invariant attributes—essential qualities—of shapes as they manipulate objects, such as the square depicted in **figure 2**. (A dynamic sketch of this problem is available at http://tinyurl.com/fig002.)



TIONS WITH EDUCATIVE DRAGGING

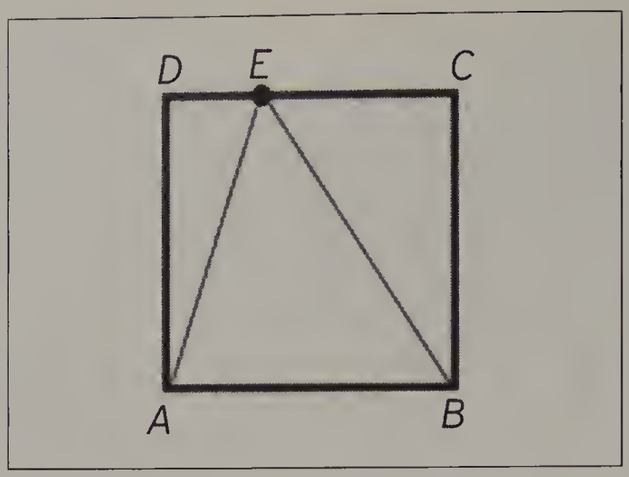


Fig. 1 The Sliding along a Side task provides students with opportunities to conjecture. Side *AB* has length 3.

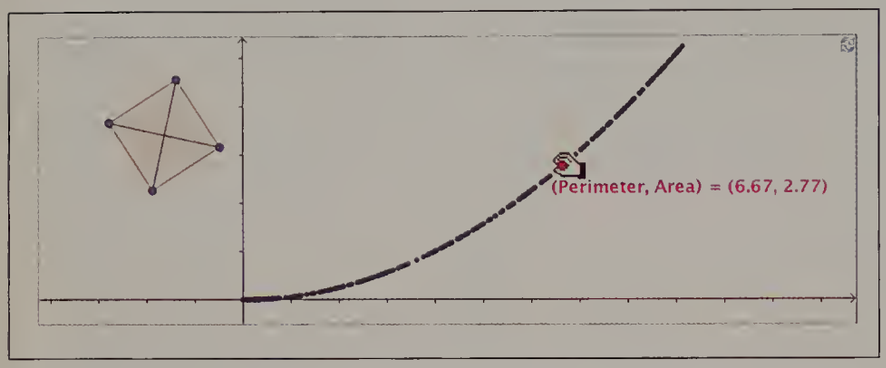


Fig. 2 This sketch shows a dynamic square with a plot of the square's area with respect to its perimeter.

As students drag vertices of a dynamic square, some notice that the diagonals of the square appear to bisect each other. Others may observe that a plot of area with respect to perimeter appears to fall on a parabolic path, regardless of the size of the square. From observations of invariance such as these, students formulate conjectures about shapes and relationships among their various attributes.

Unfortunately, the dynamic capabilities of DMS may encourage students to drag points haphazardly, giving little thought to relationships that may exist between attributes. Mindful of this tendency, we provide tasks that encourage students to predict what results they may see as a result of dragging an object before they drag the object; to reason about the results they see as they drag objects on the screen; and to reflect on the reasonableness of their predictions in light of what they observe as a result of their dragging. Such an approach, which we refer to as *educative dragging*, encourages students to develop a deeper understanding of the mathematics they encounter in DMS environments while uncovering algebraic and geometric connections within their sketches.

THE MTA PROCESS

We have developed a three-step method—Measure-Trace-Algebratize—that guides our creation of educative dragging tasks both in our research and in our teaching. The steps of the MTA process follow:

1. *Measure.* We begin by constructing a familiar shape (e.g., a rectangle, circle, or triangle) in a DMS environment. After measuring various attributes of the shape, we look for relationships among the measures (e.g., relationships between angles, side lengths, and areas) as we drag objects in our sketch.
2. *Trace.* Next, we plot an ordered pair of two seemingly related measures. For instance, we might plot the area of our dynamic shape with respect to the perimeter. Next, we examine a trace of the ordered pair as objects are dragged. Specifically, we look for traces that suggest functions.
3. *Algebratize.* Last, we use algebra (and geometric properties of shapes) to generate function models that algebraically describe the tracing of the ordered pair.

Both teachers and students may engage in the MTA process. We use the technique to build tasks for our students. Similarly, students may use the process to construct their own functions (Edwards, Phelps, and Wanko 2010).

THE SLIDING ALONG A SIDE TASK

To guide teachers through the MTA process in a professional development setting, we designed the Sliding along a Side task (see **fig. 1**).

To engage participants in the *measure* phase of the MTA process, we asked teachers to observe various measurements as point *E* is dragged and to propose various pairings that could be graphed on a coordinate plane. Teachers suggested a number of variable pairs—including the areas of the two triangles *ABE* and *ADE*, the base angles of triangle *AEB*, and the lengths of the two line segments *AE* and *BE*. Because time for exploration was limited, we opted to explore the relationship between *AE* and *BE*, the lengths of the two line segments from the moveable point to the opposite vertices of the square.

In preparation for our presentation, we had done some initial exploration of various measurement pairings and felt reasonably confident discussing a plot of *BE* with respect to *AE*. In fact, we had generated a tracing of the plot, a plot similar to the one shown in **figure 3**, generated by the sketch available at http://tinyurl.com/fig004.

We were intrigued by the tracing generated by dragging point *E* and were confident that we knew enough about the sketch and underlying measures—the distance formula, equations of circles, the Pythagorean theorem—to use the plot to highlight the MTA process at the professional development session. Needless to say, we were just as sur-

prised as everyone else by the conversation and the discovery that unfolded during our presentation.

THE "GASP!" MOMENT

To model the process of educative dragging, we asked teachers to predict the shape of the trace of BE with respect to AE without using the software. Teachers offered the following observations and conjectures:

- The two lengths are inversely related—as one length gets longer, the other gets shorter.
- The line $y = x$ will be a line of symmetry for the graph.
- The line segments BE and AE will be the same length when the triangle is isosceles.
- When the triangle is isosceles, the Pythagorean theorem can be used to find BE and AE.
- The graph will be a line with a slope of –1.
- The graph will be a curve, not a line.

To highlight the *trace* phase of the MTA process, we used GeoGebra to create a trace of the point. (GeoGebra is a freely available DMS with integrated geometry, algebra, and spreadsheet capabilities that can be downloaded from http://www.geogebra.org.) We generated a curve similar to the one shown in **figure 3**.

A number of teachers quickly hypothesized that the trace was a quarter circle. As a group, we discussed the lengths of segments AE and BE, concluding that their maximal value, $3\sqrt{2}$, can be found using the Pythagorean theorem. In this way, we found the endpoints of the trace at $(3, 3\sqrt{2})$ and $(3\sqrt{2}\ 3)$. Using geometric construction tools, we plotted the endpoints of the trace, confirming the teachers' predicted coordinates.

Next, we constructed a horizontal line through P and a vertical line through Q. The intersection of these lines was the center of the hypothesized circle. We used GeoGebra's circle tools to construct a circle that appeared to "fit" the trace of BE with respect to AE. A sketch of this construction, illustrated in **figure 4**, is available at http://tinyurl.com/qtrcirc.

The plot of the superimposed circle was convincing to everyone. One teacher asked that we derive the equation of the circle that would contain the hypothesized quarter circle. Noting that the circle was centered at $(3\sqrt{2}, 3\sqrt{2})$ with radius equaling $3\sqrt{2} - 3$, another teacher offered the derivation shown in **figure 5**.

At this point, we asked teachers to engage in the *algebratize* phase of the MTA process, generating a function relating the length of segment BE to the length of segment AE. As we walked around the room, listening to the ideas and questions being generated, several teams of teachers made the same observation: By labeling segment AE as x and applying the Pythagorean theorem twice, we can express the length of BE in terms of x, yielding a function of BE in terms of AE. After a few minutes, a volunteer shared her work with others on a projection screen (see **fig. 6**). (A pencast of **fig. 6** is available at http://tinyurl.com/figure7pencast.)

At this point, we presenters felt that we had thoroughly exhausted this example (after all, the tracing was obviously a quarter circle), but several teachers excitedly asked us to plot this function on top of the circle. Before we had a chance to plot the function on the computer, another teacher exclaimed, "The trace is *not* part of a circle!" To

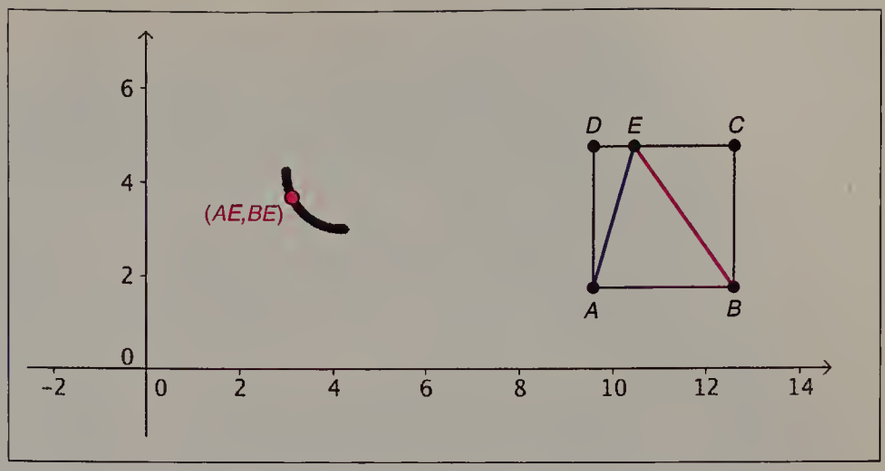


Fig. 3 This tracing was generated from the plot of BE with respect to AE.

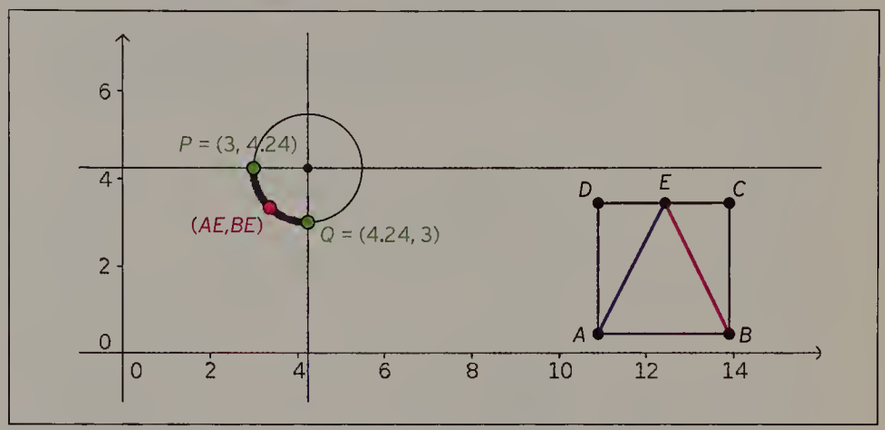


Fig. 4 The hypothesized circle is superimposed on the tracing.

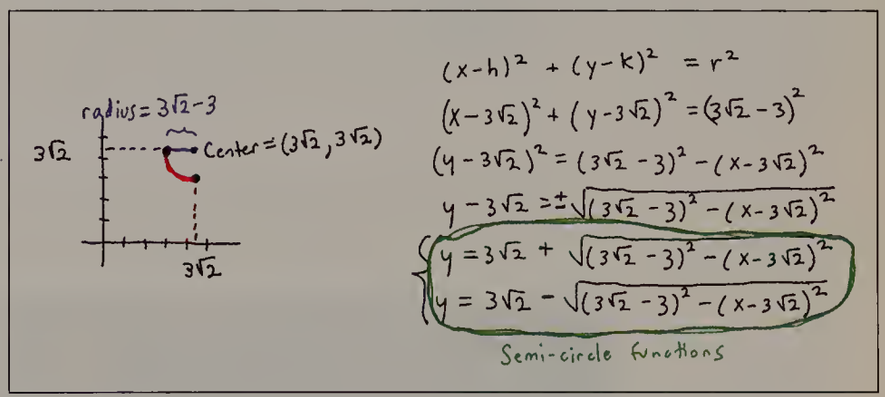


Fig. 5 One teacher derived the equations for a circle containing the hypothesized quarter circle.

foster educative dragging, we asked her to justify her claim. She suggested that the symbolic form of our graph did not match the general form of the equation of a circle. Squaring both sides makes this more clear:

$$y = \sqrt{9 + \left(3 - \sqrt{x^2 - 3^2}\right)^2}$$

$$y^2 = 9 + \left(3 - \sqrt{x^2 - 3^2}\right)^2$$

Another teacher agreed, noting that $(3 - \sqrt{x^2 - 3^2})^2$ cannot be expressed in the form $(x - h)^2$. As the group considered this, we displayed the two graphs simultaneously on the projection screen (see **fig. 7**).

The graph in **figure 7** yielded no evidence compelling enough to convince us that our earlier quarter circle conjecture was false. A portion of the magenta curve fit our quarter circle (and our plotted points) rather snugly, as illustrated in the sketch available at http://tinyurl.com/qtrcirc-func. However, the derived function was clearly not a circle. How could a circle and a noncircle simultaneously fit the plot of *BE* with respect to *AE*? This *is* strange! Another teacher stood up and said, "Good! Now zoom in!"

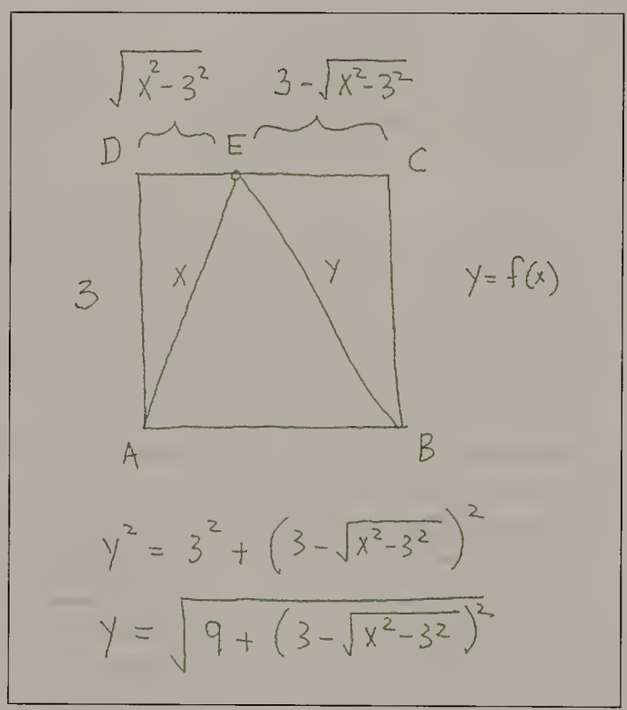


Fig. 6 One teacher derived the function showing *BE* with respect to *AE*.

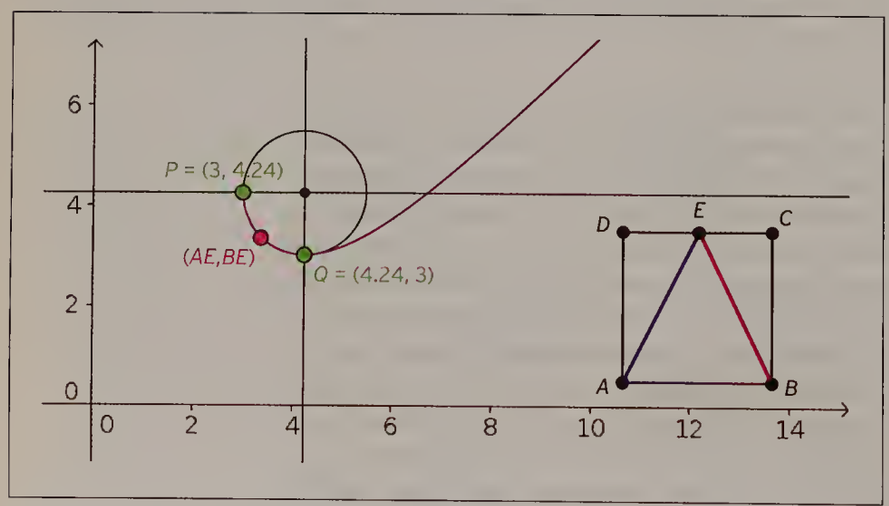


Fig. 7 We displayed the graph of the function showing *BE* with respect to *AE* (in magenta).

Up to this point, we wanted to believe that the function we had derived from our context was truly a quarter circle. As part of educative dragging, we needed to make sense of what we saw on the screen—to reason about the apparent contradictions in the sketch. When we zoomed in to look at both graphs simultaneously (see **fig. 8**), we saw a telltale sliver of space between the quarter circle (in gray) and the derived function plot (in magenta), indicating that our quarter circle conjecture was false. Equally compelling were the plotted points that lay squarely on the derived function. The group was nonplussed.

As if this moment had been planned, we reiterated the importance of listening to students' conjectures to see where they lead. In this case, a simple context led to some functions that are accessible to high school students as well as to a visually deceiving result. Clearly, the underlying mathematics that is used while solvers are algebratizing the problem is accessible to students in a typical algebra course. In fact, the Common Core State Standards (CCSSI 2010) highlight the importance of a functional approach to algebra and all mathematics, an approach demonstrated in this example. Problems like this enable students to "build functions that model relationships between two quantities" and "build new functions from existing functions"—two important CCSSM standards (CCSSI 2010, p. 56). Moreover, the algebraic derivation of the function in this example helped dispel the misconceptions that had been promulgated by the graph and helped reinforce the idea that looks can be deceiving.

But the example also shows how the MTA process can be used in a classroom to explore new problems and to encourage a problem-solving approach focused on curriculum. Although teachers are increasingly concerned about having time to cover high school curriculum, problems such as these demonstrate how they can address multiple concepts while encouraging students to strengthen their reasoning and sense-making skills.

This example barely scratches the surface of the potential in this problem and others like it. The power of this problem is found in the results of making small changes in the questions we ask:

- What would happen if we decided to change our initial assumptions regarding how we derive our

functional relationships? For example, what if, instead of applying the Pythagorean theorem to express the lengths *EA* and *EB*, we used the law of sines as a starting point?
- What would happen if we decided to explore a different variable pairing?
- What if, instead of exploring lengths *EA* and *EB*, we explore how the measures of angles *EBA* and *EAB* change as point *E* is dragged along segment *CD*?
- What would happen if we decided to explore this functional relationship in a nonsquare rectangle?

Small changes like these can lead to results even more fascinating than the one discussed here.

CONCLUSION

Using the MTA process—measuring a pair of attributes of a simple shape, tracing a plot of one measure with respect to the other, and building an algebraic model describing the trace—teachers were reminded of the powerful interplay of algebra and geometry. In addition, the importance of collaboration was clear to all. Without the observations of various group members, we might have overlooked a correct model for the Sliding along a Side task.

The use of dynamic mathematics software in our presentation did not trivialize our work with the Sliding along a Side task. We engaged in significant algebraic work throughout our investigation of the problem, connecting basic geometry (e.g., squares and right triangles) and algebra (the Pythagorean theorem) in ways not commonly encountered in school textbooks. The use of DMS encouraged us to make and test a number of conjectures by clicking and dragging; however, the software's dynamic capabilities did not preclude the need for careful observation—that is, educative dragging. Once we bought into our initial quarter circle hypothesis, careful analysis with the software ultimately challenged our collective intuition—creating cognitive dissonance that enabled us to gain new teaching and content insights.

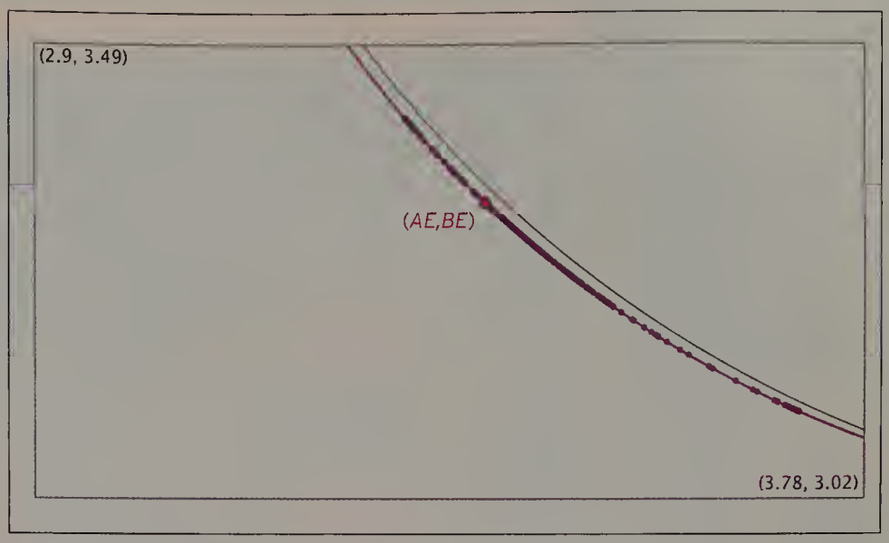


Fig. 8 We zoomed in on the plot of the quarter circle (gray) and the derived function (magenta).

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Editor's note: See also Anna E. Baccaglini-Frank, "Dragging and Making Sense of Invariants in Dynamic Geometry," *MT* April 2012 (vol. 105, no. 8, pp. 616–20).

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a surprisingly √RAD

Interesting solutions and ideas emerge when preservice and in-service teachers are asked a traditional algebra question in new ways.

Sarah D. Ledford, Mary L. Garner, and Angela L. Teachey

Sometimes, in the teaching and learning of mathematics, open-ended problems posed by teachers or students can lead to a fuller understanding of mathematical concepts—a depth of understanding that no one could have anticipated. Interesting solutions and ideas emerged unexpectedly when we asked prospective and in-service teachers an "old" algebra question in new ways. Our initial goal was to model NCTM's Process Standards in our classrooms (NCTM 2000). The result was a deeper understanding of solutions that emerge from the algorithm for solving equations involving radicals.

The old question was simply to find all solutions—true and extraneous—to radical equations. (We call equations containing radical expressions with variables in the radicand *radical equations*.) Almost every traditional high school algebra textbook includes one section devoted to solving equations of the form $\sqrt{ax+b} = cx+d$. Teachers usually instruct students to square both sides of the equation to eliminate the radical and to solve using basic algebra. Students are then told to check their answers in the original equation because extraneous solutions may appear. Often not discussed is what extraneous solutions really are and why they exist.

CHARACTERIZING SOLUTIONS OF A BASIC RADICAL EQUATION

The first new question was to characterize all possible solutions of the equation of the form $\sqrt{ax+b} = cx+d$. Goolsby and Polaski (1997) provide a graphical interpretation of the solution of $\sqrt{ax+b} = cx+d$ as the intersection of a radical function defined by the left side of the equation and a linear function defined by the right side of the equation. After some discussion of extraneous solutions, we were able to conclude that true solutions are found where the radical function intersects the line and that extraneous solutions are

ICAL problem

found where the reflection of the radical function (specifically, $y = -\sqrt{ax + b}$) intersects the line. The graphical approach provides an easy way to visualize all possible solution combinations. All possible solutions include case 1, with one true solution and one extraneous solution (see **fig. 1a**); case 2, with one true solution and no extraneous solutions (see **fig. 1b**); case 3, with one extraneous solution and no true solutions (see **fig. 1c**); case 4, with only imaginary solutions (see **fig. 1d**); case 5, with two true solutions (see **fig. 1e**); and case 6, with two extraneous solutions (see **fig. 1f**).

To characterize the solutions algebraically as well as graphically, students carried out the typical algorithm. Squaring both sides of the original equation ($\sqrt{ax + b} = cx + d$) left us with $c^2x^2 + (2cd - a)x + d^2 - b = 0$. The solution was

$$x = \frac{-2cd + a \pm \sqrt{a^2 - 4acd + 4c^2b}}{2c^2}.$$

Next, we considered the discriminant and generalized the types of solutions that we would obtain, depending on the values of a, b, c, and d. If $a^2 - 4acd + 4c^2b < 0$, then we have no true or extraneous solutions (the algebraic formulation of case 4). If $a^2 - 4acd + 4c^2b = 0$, then the solution takes the form

$$x = \frac{-2cd + a}{2c^2}.$$

In this instance, we have either one true solution or one extraneous solution (cases 2 or 3).

More specifically, we can note that we would have an extraneous solution if

$$cx + d < 0 \rightarrow c\left(\frac{-2cd + a}{2c^2}\right) + d < 0 \rightarrow \frac{a}{2c} < 0$$

(case 3). Then, using a similar argument, we would have one true solution if $a/2c > 0$. Finally, if $a^2 - 4acd + 4c^2b > 0$, we have two solutions (cases 1, 5, and 6).

A NEW QUESTION INVOLVING RADICAL EQUATIONS

We then posed the following: Write a radical equation with a true solution at $x = 5$ and extraneous solutions at $x = -1$ and $x = -2$. Asking an open-ended question about these radical equations seemed like a logical next step. We were surprised and pleased to find that the problem was more difficult than we had expected and led to some interesting solution strategies.

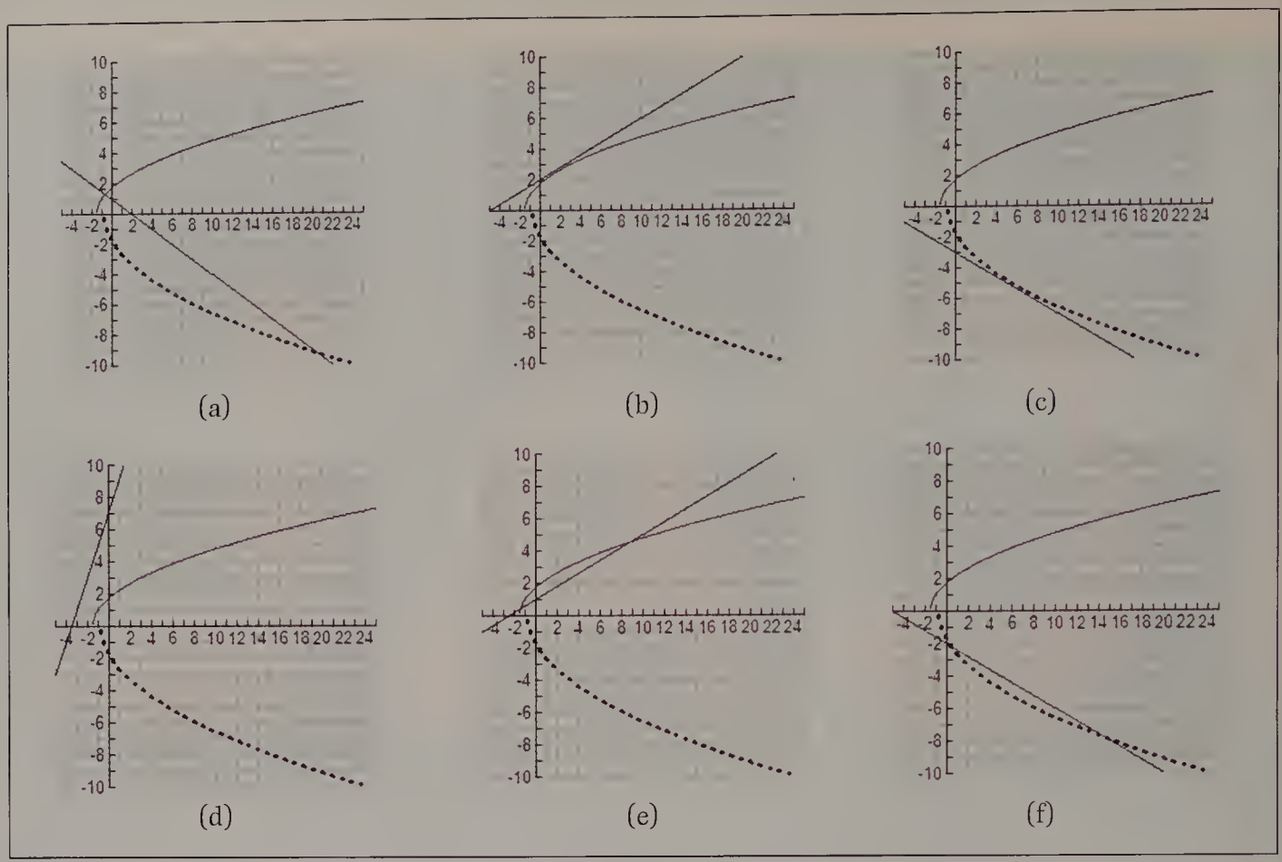


Fig. 1 These graphs (**a**-**f**) indicate all possible cases.

The new question led us to a discussion of the fundamental theorem of algebra and to a discussion of one-to-one functions. Note that an equation with three real solutions (extraneous or not) cannot be of the form $\sqrt{ax+b} = cx + d$ because the quadratic produced by squaring both sides of the equation has exactly two solutions. This is a basic application of the fundamental theorem of algebra.

Because we need three solutions, a typical first impulse in solving the new problem would be to consider a radical equation that would produce a cubic instead of a quadratic, an equation of the form $\sqrt[3]{ax+b} = cx + d$, for example. However, cubing both sides of this equation will not produce extraneous solutions because the basic cubic function $y = x^3$ is a one-to-one function.

Recall the definition of one-to-one functions: A function f is one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ (where x_1 and x_2 are in the domain of f). When we solve any equation of the form $\sqrt{ax+b} = cx + d$, we can square both sides of the equation $f(\sqrt{ax+b}) = f(cx+d)$, but because the function is not one-to-one, we cannot then say that $\sqrt{ax+b} = cx + d$. When we solve any equation of the form $\sqrt[3]{ax+b} = cx + d$, we can cube both sides and be confident that $f(\sqrt[3]{ax+b}) = f(cx+d)$ does imply that $\sqrt[3]{ax+b} = cx + d$.

Method 1: Exploiting the Sign of the Extraneous Roots

We were thrilled to find that when students were unable to solve the problem, many used algebraic methods to solve a simpler problem that involved only two solutions, hoping that their technique could be generalized to the more difficult problem. They decided to find a radical equation that had one true solution at $x = 5$ and one extraneous solution at $x = -1$. Starting with the end in sight, $(x - 5) \cdot (x + 1) = 0$, which is equivalent to $x^2 - 4x - 5 = 0$, one student simply transformed this equation to $x = \sqrt{4x+5}$. Using technology, we can check for the solutions of this as we graph $y_1 = x$, $y_2 = \sqrt{4x+5}$ (which will show the true solution), and $y_3 = -\sqrt{4x+5}$ (which will show the extraneous solution).

This method was then extended to the problem at hand. Again, we started with the end in sight: $(x - 5)(x + 1)(x + 2) = 0$, which is equivalent to $x^3 - 2x^2 - 13x - 10 = 0$. We solved for

$$x^2 = \frac{x^3 - 13x - 10}{2}$$

so that we could take the square root of both sides, leaving

$$x = \pm\sqrt{0.5x^3 - 6.5x - 5}.$$

In graphing, we can see that this approach does, in fact, yield the desired solutions (see **fig. 2**).

Note that the method described here works only because the extraneous solutions we chose were negative; in other words, a negative value cannot be equal to a principal square root. What if we wanted

all solutions to be positive? A well-chosen horizontal shift of both sides of the equation

$$x = \sqrt{0.5x^3 - 6.5x - 5}$$

would produce

$$x - 4 = \sqrt{0.5(x-4)^3 - 6.5(x-4) - 5},$$

an equation that has extraneous solutions at $x = 2$ and $x = 3$ and a true solution at $x = 9$.

Method 2: An Algebraic Approach

A second approach is similar to the first yet quite elegantly accounts for the negative extraneous solutions and the positive true solutions. The student who came up with this approach also attempted the simpler problem first (a true solution at $x = 5$ and one extraneous solution at $x = -1$). She carefully selected a perfect square trinomial $(2x - 3)^2 = 4x^2 - 12x + 9$ so that $2x - 3 < 0$ for $x = -1$ (the extraneous solution) and $2x - 3 > 0$ for $x = 5$ (the true solution). Next, she added the perfect square trinomial to both sides of the equation $x^2 - 4x - 5 = 0$ to obtain $5x^2 - 16x + 4 = (2x - 3)^2$. She then took the square root of both sides—

$$\sqrt{5x^2 - 16x + 4} = 2x - 3$$

—and achieved a radical equation with the desired solutions.

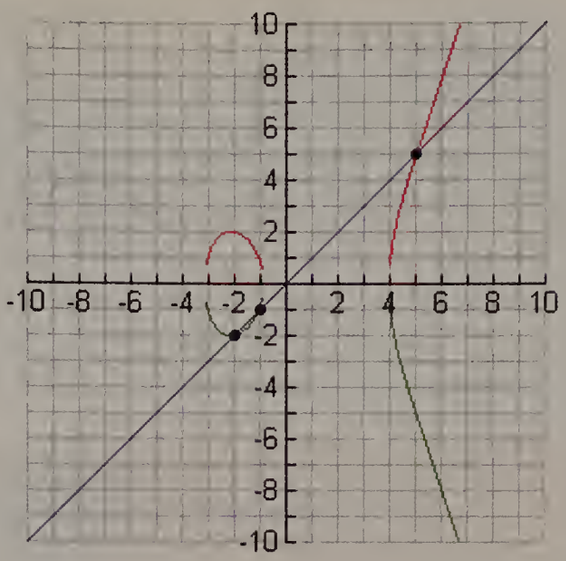
In extending this method to the problem with three solutions, we started with the same cubic as before: $x^3 - 2x^2 - 13x - 10 = 0$. We then carefully selected a perfect square, $(x - 3)^2$, to ensure that $x - 3 < 0$ for $x = -1$ and $x = -2$ (the extraneous solutions) and that $x - 3 > 0$ for $x = 5$ (the true solution). After adding the squared binomial to both sides of the equation, we had $x^3 - x^2 - 19x - 1 = (x - 3)^2$ and, finally,

$$\sqrt{x^3 - x^2 - 19x - 1} = x - 3.$$

Again, in graphing each side of this equation and the reflection of the radical in the x-axis, we had the desired solutions as the intersections.

Method 3: A Graphical Approach

A few students used a graphical approach for the simpler problem. They started with a function such as $y = \sqrt{x + 2}$ and then determined a line through the points $(-1, -1)$ and $(5, \sqrt{7})$. These points were obtained using the same ideas as the previous two simpler solutions because we wanted the true solution to be at $x = 5$ and the extraneous solution at $x = -1$. Therefore, students using this method substituted $x = 5$ into their radical equation, producing



The graph of

$$y = x \text{ and}$$

$$y = \pm\sqrt{0.5x^3 - 6.5x - 5}$$

shows a true solution at $x = 5$ and extraneous solutions at $x = -1$ and $x = -2$.

Fig. 2 The true solution is at the intersection of the red curve with the line; extraneous solutions are at the intersection of the green curve with the line.

the value $\sqrt{7}$. When substituting $x = -1$, they got an output of $+1$; however, because this solution is meant to be the extraneous one, it would have an output of -1, according to the radical equation's reflection. The line passing through the two points is

$$y + 1 = \frac{\sqrt{7} + 1}{6}(x + 1),$$

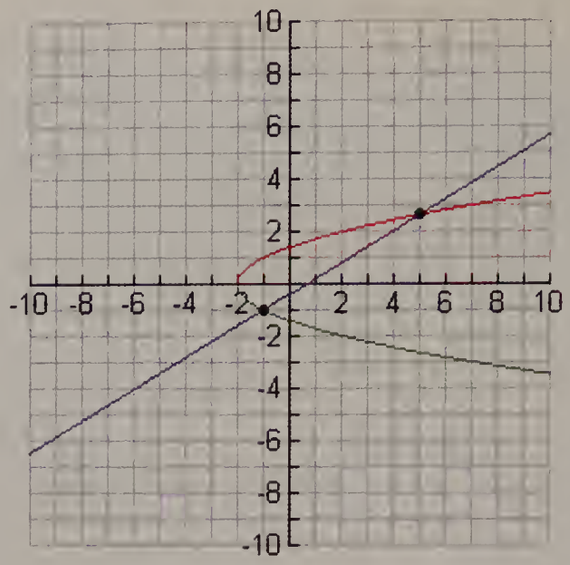
leading to the equation

$$\sqrt{x + 2} = \frac{\sqrt{7} + 1}{6}x + \frac{\sqrt{7} - 5}{6}.$$

Thus, we arrive at the correct solutions (see **fig. 3**).

This method finally led us to a more general approach that will work for any number of true and extraneous solutions. This approach involved obtaining an equation of the form $f(x) = \pm\sqrt{g(x)}$ such that $f(-1) = -\sqrt{g(-1)}$, $f(-2) = -\sqrt{g(-2)}$ (to ensure extraneous solutions), and $f(5) = \sqrt{g(5)}$ (the true solution). The next step was to arbitrarily select "nice" values so that $g(-1) = 1$, $g(-2) = 4$, and $g(5) = 9$, leading to $f(-1) = -1$, $f(-2) = -2$, and $f(5) = 3$.

Therefore, we need a function g whose graph contains the points $(-1, 1)$, $(-2, 4)$, and $(5, 9)$ and a function f whose graph contains the points $(-1, -1)$, $(-2, -2)$, and $(5, 3)$. Because we had three points

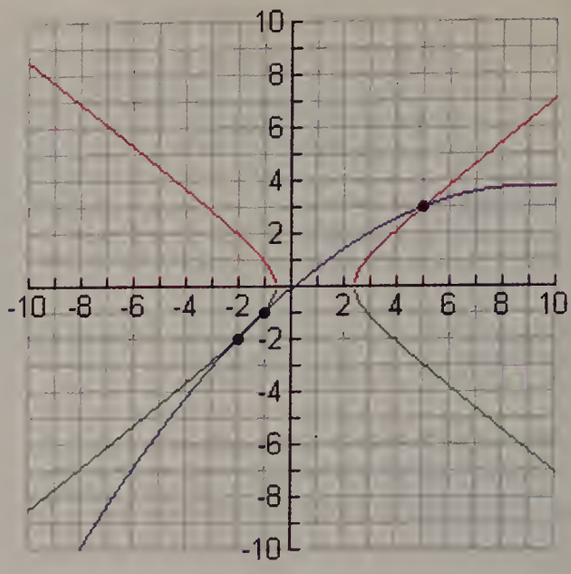


The graph of

$$\pm\sqrt{x+2} = \frac{\sqrt{7}+1}{6}x + \frac{\sqrt{7}-5}{6}$$

shows a true solution at $x = 5$ and extraneous solutions at $x = -1$.

Fig. 3 This graph shows a true solution at the intersection of the red curve and the blue curve and an extraneous solution at the intersection of the green curve and the blue curve.



The graph of

$$-\frac{1}{21}x^2 + \frac{6}{7}x - \frac{2}{21} = \pm\sqrt{\frac{13}{21}x^2 - \frac{8}{7}x - \frac{16}{21}}$$

shows a true solution at $x = 5$ and extraneous solutions at $x = -1$ and $x = -2$.

Fig. 4 This graph shows a true solution at the intersection of the red curve and the blue curve and extraneous solutions at the intersections of the green curve and the blue curve.

for each function, we concluded that we could calculate quadratic functions for $f(x)$ and $g(x)$.

We then substituted our points into $f(x) = ax^2 + bx + c$ to generate the two systems of equations below:

$$\begin{aligned} f(x) \rightarrow a(-1)^2 + b(-1) + c &= -1 \\ a(-2)^2 + b(-2) + c &= -2 \\ a(5)^2 + b(5) + c &= 3 \end{aligned} \qquad \begin{aligned} g(x) \rightarrow a(-1)^2 + b(-1) + c &= 1 \\ a(-2)^2 + b(-2) + c &= 4 \\ a(5)^2 + b(5) + c &= 9 \end{aligned}$$

We obtained the following functions—

$$f(x) = -\frac{1}{21}x^2 + \frac{6}{7}x - \frac{2}{21}$$

and

$$g(x) = \frac{13}{21}x^2 - \frac{8}{7}x - \frac{16}{21},$$

—leading to the equation

$$-\frac{1}{21}x^2 + \frac{6}{7}x - \frac{2}{21} = \pm\sqrt{\frac{13}{21}x^2 - \frac{8}{7}x - \frac{16}{21}}.$$

Graphically, we can see that the solutions are indeed those that we were looking for (see **fig. 4**).

Exploiting a Different Source of Extraneous Roots

Another group of students had been exploring equations such as

$$\frac{\left(x^2 + x - 2\right)}{x+2} = 4.$$

Applying the usual algorithm for solving the equation produces the solutions $x = -2$, which is clearly an extraneous solution, and $x = 5$. Note that $x + 2$ is a factor of the numerator and the denominator of the left side of the equation. To produce a true solution at $x = 5$, we begin with $x - 1 = 4$. To produce extraneous solutions at $x = -1$ and $x = -2$, we then form a rational expression on the left side of the equation:

$$\frac{(x-1)(x+1)(x+2)}{(x+1)(x+2)} = 4$$

This time the extraneous roots come from holes in the domain of the rational expression on the left side of the equation.

As a class, we also discussed other types of functions that are not one-to-one and that could therefore produce extraneous solutions when applied to both sides of an equation—trigonometric functions, for example.

CONCLUSION

Asking open-ended questions about an old problem led to a surprising variety of solutions and solution strategies that involved algebraic, tabular, and graphical representations of functions. The characterization of the solutions of the equation $\sqrt{ax+b} = cx + d$ illustrates the power of a graphical approach, whereas the algebraic approach makes specific the relationship among the parameters a, b, c, and d. Asking students to produce an equation with certain extraneous and true solutions led to a synthesis of many concepts, including the fundamental theorem of algebra, one-to-one functions, the problem-solving strategy of solving a simpler problem, horizontal shifts of functions, solutions of systems of equations, and functions with holes in their domains. Most important, we were able to engage students in an exploration of radical equations with true and extraneous solutions without simply relying on an algorithm that they had been taught. These investigations ultimately led them to a deeper understanding of the mathematical concepts.

As teachers, we gained new insights from students' ideas and strategies. Classroom discussions can be rich and powerful when students are encouraged to explore mathematics.

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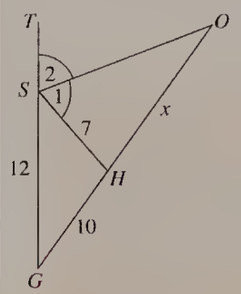
The Mathlete team is exactly 37.5% female. If no additional boys join the team, what is the smallest number of girls who can join the team to raise the percentage to exactly 45%?

1

In $\triangle GHS$, $SG = 12$, $GH = 10$, and $HS = 7$. Side GS is extended through S to point T, and the angle bisector of $\angle HST$ intersects extended side GH at O. Find HO.



2

The probability that statement P is true is 3/7, and the probability that statement Q is true is 5/12. Determine the probability that $P \rightarrow Q$ is true.

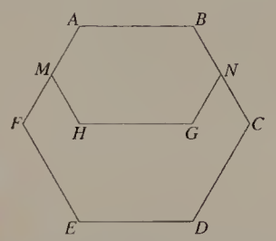
3

Find the square root of the sum of the squares of the roots of the following:

$$\left(9x^2-4\right)^4-10\left(9x^2-4\right)^2+9=0$$

4

ABCDEF is a regular hexagon with M the midpoint of $\overline{AF}$ and N the midpoint of $\overline{BC}$. Hexagon *ABNGHM* is formed by reflecting $\overline{AM}$, $\overline{AB}$, and $\overline{BN}$, over line MN. Find the ratio of the area of *ABCDEF* to the area of *ABNGHM*.



5

In a circle, a central angle of 120° intercepts arc AB. The region between minor arc AB and line segment AB is called an *arc segment* of the circle. Find the ratio of the area of the arc segment to the area of the circle.

6

Find the dimensions of two cubes with integral edges such that the sum of their volumes is numerically equal to the sum of the lengths of all their edges.

7

Columbus underestimated the circumference of Earth by about 25%. The radius of Earth is between 3947 and 3968 miles. Columbus left Palos, Spain, on August 3, 1492, and arrived in the Bahamas on October 12, 1492. How many extra days did he spend at sea as a result of his error?

8

In 1000 CE, Leif Erikson was the first European to arrive in North America, and Leif Erikson Day is always celebrated on October 9. Columbus Day is currently celebrated on the second Monday in October. What is the probability that the two holidays fall on the same day?

9

A brother and a sister each had some pennies. After his mother gave him 3 more pennies, the brother gave half his pennies to his sister. He repeated sharing with his sister after his father and then his grandmother each gave him 3 additional pennies. His sister now had 5 times the number of pennies that he had. What is the maximum number of pennies that the brother had initially?

10

Each digit in the multiplication table represents a digit other than itself. Create a new table, placing the digits in their correct places.

	1	2	4	8
1	31	71	51	29
2	71	4	30	76
4	51	30	67	15
8	29	76	15	28

11

The side lengths of a rectangular picture are integers. Compute the area of the largest picture that can be framed using a 1-in.-wide frame such that the area of the picture equals the area of its frame.

In rectangle *ABCD*, P is on $\overline{AD}$, 6 units from A and 14 units from D. Q is on $\overline{CD}$, 7 units from D and 6 units from C. R is on $\overline{BC}$, 8 units from C; and S is on $\overline{AB}$, 5 units from B. From which of the four points—P, Q, R, and S—is the sum of the distances to the other three points a minimum?

Find the dimensions of a rectangular solid for which the sum of all the edges is 76, the total surface area is 206, and the volume is 165.

Simplify the following expression:

$$\frac{\sqrt{2}+\sqrt{10}}{2\sqrt{3+\sqrt{5}}}$$

Two sequences of consecutive positive integers both have a sum of 50. Find the sum of the four numbers that begin and end the two sequences.

16

How many sequences of consecutive positive integers have a sum of 30?

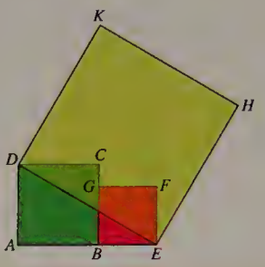
17

Determine the smallest prime number that divides the following sum:

$$3^{12} + 5^{13} + 7^{14} + 11^{15}$$

18

$ABCD$, $BGFE$, and $EDKH$ are all squares. Compute the ratio AB/BE that will maximize the ratio of the area of $EDKH$ to the sum of the areas of the other two squares.



19

Solve for all real values of x:

$$\frac{\left(x^2 - 13x + 40\right)\left(x^2 - 13x + 42\right)}{\sqrt{x^2 - 12x + 35}} = 0$$

20

For all x and y,

$$f(x) - f(y) = f(xy).$$

Calculate the value of

$$f(2012).$$

21

A circle is inscribed in a right triangle with point P common to both the circle and hypotenuse AB. Find the area of the triangle if $AP \cdot PB = 24$.

22

Determine the number of right triangles that can be constructed with integer sides such that the radius of the inscribed circle is 4.

23

For United Nations Day (October 24), the following contest was held: Given a rectangular banner 4 ft. × 9 ft., cut it into two pieces that could be sewn together to form a new banner 3 ft. × 12 ft. How would you accomplish this task and thus win the contest?

24

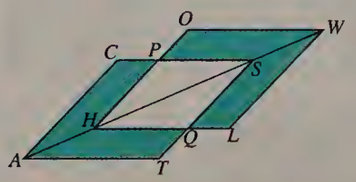
In $\triangle RST$,

$$\sin R = 3/5 \text{ and } \sin S = 15/17.$$

Find $\sin T$.

25

Two congruent rhombi, $CATS$ and $HOWL$, overlap so that their longer diagonals lie on the same line. The area common to both rhombi equals the area of each region shaded green. Find the ratio PS/CP.



26

A millionaire decided to give $20 to each trick-or-treater wearing a homemade costume and $10 to each trick-or-treater wearing a store-bought costume. Fifty percent of the children offered $20 refused to accept it, whereas all the children offered $10 took the money. If 85 children accepted the treat, what was the difference between the maximum and the minimum amounts that the millionaire could dispense?

27

Two mothers and their two daughters went shopping. Each spent the same amount of money, and the total spent was $243. If the amount each spent was a whole number of dollars, how was this possible?

28

On Halloween morning, a store owner decided to sell his last two costumes for $18 each. He made a profit of 25% on one outfit and took a 20% loss on the other outfit. What was his total profit or loss for these two sales?

29

The cost of 2 large pumpkins plus 1 medium-sized pumpkin is the same as the cost of 8 small pumpkins. One large pumpkin costs the same as 1 medium-sized pumpkin plus 1 small pumpkin. How many small pumpkins can you buy for the price of a large pumpkin?

30

Given the parametric equations

$$x = 2a\tan\theta \text{ and } y = 2a\cos^2\theta,$$

eliminate the parameter (θ) and solve the resulting equation for y in terms of a and x.

31

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SOLUTIONS to calendar

Looking for more Calendar problems?
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Edited by ***Margaret Coffey,*** *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and* ***Art Kalish,*** *artkalish@verizon.net, Syosset High School (retired), Syosset, NY 11791*

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

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1. 12. Let g be the number of additional girls and let $3t$ be the number of girls on the team. Since 37.5% = 3/8, we have the following:

$$\frac{3t+g}{8t+g}=\frac{45}{100}=\frac{9}{20}\rightarrow$$
$$20\left(3t+g\right)=9\left(8t+g\right)\rightarrow$$
$$12t=11g$$

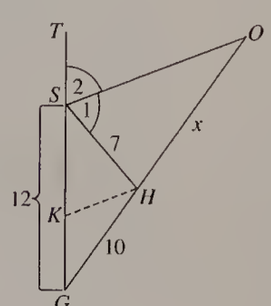
The least value of g that allows t to be an integer is 12.

2. $HO = 24$. Draw a line parallel to $\overline{OS}$ through point H. Label the line's intersection with $\overline{GS}$ as K. Since alternate interior and corresponding angles of parallel lines are congruent, $\angle SKH \cong \angle 2$, and $\angle SHK \cong \angle 1$. We know that $\angle 1 \cong \angle 2$, so $\angle SKH \cong \angle SHK$, and $\triangle SHK$ is isosceles. This means that $SK = 7$ and $KG = 5$. Since $\overline{KH} \parallel \overline{OS}$,

$$\frac{GK}{KS}=\frac{GH}{HO}\rightarrow\frac{5}{7}=\frac{10}{x}.$$

Thus, $HO = 14$.

Alternate solution: The problem can also be solved by using the law of cosines to find the measures of the angles of $\triangle GHS$ and then applying the law of sines to $\triangle HOS$.



3. 3/4. The statement $P \rightarrow Q$ is true when it is not false. The statement is false whenever P is true and Q is false. Since the probability that Q is false = 1 – 5/12 = 7/12, the probability that $P \rightarrow Q$ is false is the product $(3/7) \cdot (7/12) = 3/12 = 1/4$. The probability that $P \rightarrow Q$ is true = 1 – 1/4 = 3/4.

4. 4/3. Let $y = (9x^2 - 4)^2$. Then the equation becomes $y^2 - 10y + 9 = 0 \rightarrow y = 9$ or $y = 1$. Then

$$\left(9x^2-4\right)^2=9\vee\left(9x^2-4\right)^2=1$$
$$\left(9x^2-4=3\right)\vee\left(9x^2-4=-3\right)$$
$$\vee\left(9x^2-4=1\right)\vee\left(9x^2-4=-1\right)\rightarrow$$
$$\left(\left(x^2=\frac{7}{9}\right)\right)\vee\left(\left(x^2=\frac{1}{9}\right)\right)\vee\left(\left(x^2=\frac{5}{9}\right)\right)$$
$$\vee\left(\left(x^2=\frac{3}{9}\right)\right)$$

Therefore, the sum of the squares of the roots is

$$\frac{7}{9}+\frac{1}{9}+\frac{5}{9}+\frac{3}{9}=\frac{16}{9},$$

and the square root of 16/9 is 4/3.

5. 12/5. The area of a regular hexagon = $ap/2$ where a is the apothem and p is the perimeter. If a side of the hexagon is s, the perimeter is $6s$. The apothem is the line segment drawn from the center of the hexagon perpendicular to one of its sides. Since the radii of a hexagon form equilateral triangles of side s, the apothem is equal to the altitude of one of these triangles, or $s\sqrt{3}/2$. Thus,

$$A_{ABCDEF}=\frac{\left(s\sqrt{3}/2\right)6s}{2}=\frac{3s^2\sqrt{3}}{2}.$$

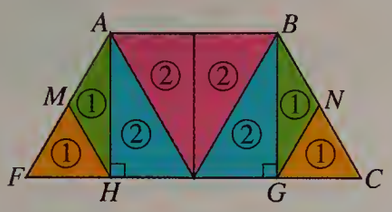
The area of the smaller hexagon can be found by subtracting the areas of $\triangle FMH$ and $\triangle GNC$ from half the area of the big hexagon. Therefore,

$$A_{ABNGHM} = \frac{1}{2} \cdot \frac{3s^2\sqrt{3}}{2} - 2\left(\frac{s^2\sqrt{3}}{16}\right) = \frac{5s^2\sqrt{3}}{8}.$$

The ratio of the two areas equals

$$\left(\frac{3s^2\sqrt{3}}{2}\right) \div \left(\frac{5s^2\sqrt{3}}{8}\right) = \frac{12}{5}.$$

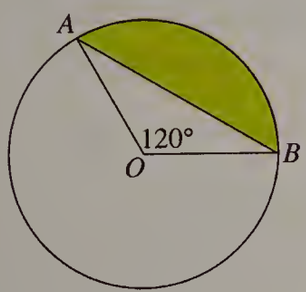
Alternate solution: This is a derivation without words. The circled numbers in each region represent their relative areas. Therefore, the ratio of the areas of the two hexagons equals $2(1 + 1 + 2 + 2 + 2 + 2 + 1 + 1)/(1 + 2 + 2 + 2 + 2 + 1) = 24/10 = 12/5$.



6. 0.1955. The area of the arc segment can be found by subtracting the area of $\triangle AOB$ from the area of sector AOB. If the radius of the circle is r, then

$$\frac{A_{segAB}}{A_{circle}} = \frac{\frac{\pi r^2}{3} - \left(\frac{1}{2}\right)\left(r^2 \sin 120°\right)}{\pi r^2}$$

$$= \frac{4\pi - 3\sqrt{3}}{12\pi} \approx 0.1955.$$



7. The edges of the cubes are 2 and 4. Let a and b represent the edge lengths. Then $a^3 + b^3 = 12(a + b)$. Factoring $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ results in $a^2 - ab + b^2 = 12 \rightarrow a^2 - 2ab + b^2 = 12 - ab = (a - b)^2 = 12 - ab$. Therefore, $12 - ab$ must be a perfect square. The only perfect square that satisfies the equation is 4; therefore, $ab = 8$, $a = 4$, and $b = 2$ ($a = 1$ and $b = 8$ does not work).

8. 17.75 days. Let d represent the number of days that Columbus expected the trip to take. Calculate the number of days from August 3 to October 12 to be 71. Solve the equation $4d/3 = 71$ to find that Columbus expected the trip to take 53.25 days, which is 17.75 days fewer than the actual trip.

9. 1/7. The second Monday in October can be any day from October 8 to October 14, so there is a 1 in 7 probability that Columbus Day will fall on October 9.

10. 23. Let x = the number the brother had and y = the number the sister had. Then, after the first round of receiving and sharing the pennies, the boy had

$$\frac{x+3}{2} \text{ pennies,}$$

and the girl had

$$y + \frac{x+3}{2} \text{ pennies.}$$

After his father gave him 3 pennies and he gave half his total to his sister, the brother had

$$\frac{1}{2}\left(\frac{x+3}{2} + 3\right) = \frac{x+9}{4} \text{ pennies,}$$

and the sister had

$$y + \frac{x+3}{2} + \frac{x+9}{4} = y + \frac{3x+15}{4} \text{ pennies.}$$

After his grandmother gave him 3 pennies and he gave half his total to his sister, the brother had

$$\frac{1}{2}\left(\frac{x+9}{4} + 3\right) = \frac{x+21}{8} \text{ pennies,}$$

and she had

$$y + \frac{3x+15}{4} + \frac{x+21}{8} = y + \frac{7x+51}{8} \text{ pennies.}$$

Set

$$y + \frac{7x+51}{8} = 5\left(\frac{x+21}{8}\right)$$

and simplify to

$$y = \frac{54 - 2x}{8}.$$

The largest integral value for x that maintains a positive integer value for y is 23, the greatest number of pennies the brother could have had initially.

11. The key is to realize that the main diagonal of the array consists of perfect squares. Therefore, 4 must be a single-digit perfect square. Since 4 cannot be 0, 1, or 4, it must be 9. This result requires 67 to represent 81, so 6 must be 8, and 7 must be 1. Continue in this fashion to complete the table:

Given	0	1	2	3	4	5	6	7	8	9
Correct	7	5	3	2	9	4	8	1	6	0

The resulting multiplication table follows:

	5	3	9	6
5	25	15	45	30
3	15	9	27	18
9	45	27	81	54
6	30	18	54	36

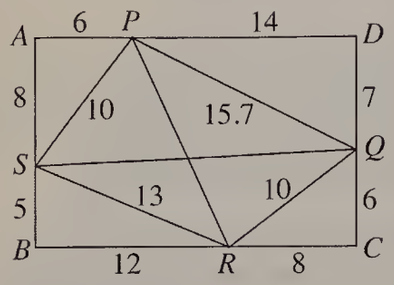
12. The maximum area of the picture is 30. Two possible rectangular pictures can be framed with a 1-in.-wide frame such that the frame has the same area as the picture. Let the width of the picture be x, and let the length be y. Then we have the following:

$$xy = (x + 2)(y + 2) - xy \rightarrow$$
$$xy = 2x + 2y + 4 \rightarrow$$
$$xy - 2y = 2x + 4 \rightarrow$$
$$y = \frac{2x+4}{x-2}$$

The only integers that satisfy this equation are (3, 10) and (4, 6). If the width is 3 and the length is 10, the area is 30.

13. Point R. The lengths of the four sides of quadrilateral $PQRS$ are $PS = 10$, $SR = 13$, $RQ = 10$, and $QP = \sqrt{245} \approx 15.7$. It should be obvious that the smallest sum stems from either R or S. To make the final decision, consider that

$PR < AR \approx 17.69$ and that $QS > AD = 20$. Therefore, $PR < QS$, and the smallest sum comes from point R.



14. 3, 5, 11. Let l, w, and h be the length, width, and height of the solid. Then $4(l + w + h) = 76$; $2(lw + wh + lh) = 206$; and $lwh = 165$. Solve the first-degree equation for $w + h = 19 - l$. Factor the second-degree equation to get $l(w + h) + wh = 103$. Solve the third-degree equation for $wh = 165/l$. Substitute the first and third results into the factored second-degree equation to get $l(19 - l) + 165/l = 103$. This cubic equation can be written as $l^3 - 19l^2 + 103l + 165 = 0$. If the roots of this equation are rational, then the solutions must be factors of 165 that add to 19. Since 165 factors into $3 \cdot 5 \cdot 11$, which happen to add to 19, these factors are the dimensions of the rectangle. An interesting observation gives us a clue to a quicker way to find the final cubic equation. In a generalized third-degree polynomial, when the leading coefficient is 1, the coefficient of the quadratic term is the opposite of the sum of the roots, the coefficient of the linear term is the sum of the products of the roots taken two at a time, and the constant term is the opposite of the product of the roots. In other words, $x^3 - (l + w + h)x^2 + (lw + wh + lh)x - lwh = 0$.

15. 1. Let

$$x = \frac{\sqrt{2} + \sqrt{10}}{2\sqrt{3 + \sqrt{5}}}$$

and then square both sides of the equation. Therefore,

$$x^2 = \left(\frac{\sqrt{2} + \sqrt{10}}{2\sqrt{3 + \sqrt{5}}}\right)^2 = \frac{2 + 2\sqrt{20} + 10}{4\left(3 + \sqrt{5}\right)}$$

$$= \frac{12 + 4\sqrt{5}}{12 + 4\sqrt{5}} = 1.$$

Since $x^2 = 1$ and $x > 0$, $x = 1$.

16. 45. The sum of two consecutive positive integers must be odd, so the sequence must be longer than two numbers. Since the sum of three consecutive positive integers is a multiple of 3 and 50 is not, the sequence must have at least four integers. Solve $x + (x + 1) + (x + 2) + (x + 3) = 50$ to obtain $x = 11$; the sequence is 11, 12, 13, and 14. Solving a similar equation for five consecutive positive integers yields the sequence 8, 9, 10, 11, 12. Therefore, the answer is the sum of $8 + 12 + 11 + 14 = 45$.

17. 3. Proceeding as above, we find that it is possible for three consecutive positive integers to sum to 30 because 30 is a multiple of 3. The three sequences are 9, 10, 11; 4, 5, 6, 7, 8; and 6, 7, 8, 9.

18. 2. Each individual term is odd, so the sum is even. Therefore, the smallest prime divisor is 2.

19. 1.61803. Let $BE = 1$ and $AB = x$. Apply the Pythagorean theorem to find that $DE^2 = (x + 1)^2 + x^2$. We are looking to maximize the expression

$$\frac{A_{EDKH}}{A_{ABCD} + A_{BGFE}} = \frac{(x+1)^2 + x^2}{x^2 + 1^2}.$$

The value of x that will maximize this function can be found by using calculus or by graphing the function on a calculator and using the **find max** operation. The value of x that satisfies the condition is the golden ratio = phi = $\phi \approx 1.61803$.

20. 8. Multiply both sides of the equation by the denominator and then factor the left side. The result is $(x - 5)(x - 8)(x - 6) \cdot (x - 7) = 0$. The four possible solutions are 5, 6, 7, and 8. However, when these values are substituted into the original equation, 5 and 7 make the denominator 0, and the number 6 makes the denominator imaginary. Thus, the only solution is 8.

21. 0. If we let $x = 0$, we see that $f(0) - f(y) = f(0)$. Therefore, for all values of y, $f(y) = 0$, so $f(2012) = 0$.

22. 24. The product of the segments of the hypotenuse formed by the point of tangency of the inscribed circle is always equal to the area of the right triangle. To prove this, let a and b be the segments of the hypotenuse. The legs of the triangle will be $a + x$ and $b + x$. Then we have the following:

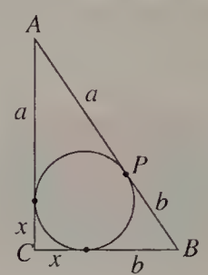
$$(a+x)^2 + (b+x)^2 = (a+b)^2$$
$$a^2 + 2ax + x^2 + b^2 + 2bx + x^2 = a^2 + 2ab + b^2$$
$$2x^2 + 2ax + 2bx = 2ab$$
$$x^2 + ax + bx = ab$$

The area of the triangle is

$$A = \frac{1}{2}(a+x)(b+x) = \frac{1}{2}(ab + ax + bx + x^2).$$

By substitution,

$$A = \frac{1}{2}(ab + ab) = ab = 24.$$



23. 3. There are only three such triangles: 9-40-41, 10-24-26, and 12-16-20. Several formulas relate the sides of a right triangle to the radius of the inscribed circle. One such formula is

$$r = \frac{a + b - c}{2}$$

where c is the hypotenuse and a and b are the legs. Apply this formula with $r = 4$ to get $a + b - c = 8$. Since

$$c = \sqrt{a^2 + b^2},$$

we have

$$a + b - \sqrt{a^2 + b^2} = 8 \rightarrow$$
$$\sqrt{a^2 + b^2} = (a + b) - 8.$$

Square both sides of the equation and solve for a:

$$a^2 + b^2 = a^2 + b^2 + 64 + 2ab - 16a - 16b \rightarrow$$
$$0 = 32 + ab - 8a - 8b$$

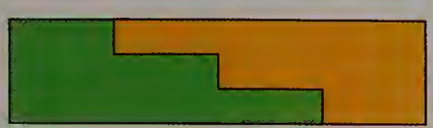
so

$$\therefore \quad a = \frac{32 - 8b}{8 - b} = 8 + \frac{32}{b - 8}.$$

Since a and b are integers, $b - 8$ must be a factor of 32. The factors of 32 are 1, 2, 4, 8, 16, and 32. Let $b - 8$ take on each of these values and solve for a. The results in the form (b, a) are (9, 40), (10, 24), (12, 16), (16, 12), (24, 10), and (40, 9), creating three unique right triangles.

24. Divide the height and width into three equal parts and then slide one piece, as shown in the diagram.

(a)



(b)

25. 84/85. Since angles R, S, and T sum to 180°, $T = (180 - (R + S))°$, and $\sin T = \sin(180 - (R + S))°$. Using the expansion of the sine of a difference or applying transformation rules, we have $\sin(180 - x)° = \sin x°$, so $\sin T = \sin(R + S) = \sin R \cdot \cos S + \sin S \cdot \cos R$. We can calculate the two cosine expressions by using the formula

$$\sin x = \sqrt{1 - \cos^2 x}$$

or by recognizing the Pythagorean triples 3-4-5 and 8-15-17. Therefore, $\sin T = (3/5)(8/17) + (15/17)(4/5) = 84/85$.

26. $\sqrt{2} + 1$. $PHQS$ is a rhombus similar to the original rhombi and half the area of each. Since the ratio of the areas of similar shapes equals the ratio of the squares of a pair of corresponding lengths, we have the following:

$$\frac{A_{PHQS}}{A_{CATS}} = \frac{(PS)^2}{(PS + CP)^2} = \frac{1}{2} \rightarrow$$

$$PS\sqrt{2} = PS + CP \rightarrow$$

$$PS\sqrt{2} - PS = CP \rightarrow$$

$$PS(\sqrt{2} - 1) = CP \rightarrow$$

$$\frac{PS}{CP} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$$

27. \$850. If no trick-or-treaters wore a homemade outfit, then the millionaire would give out 85 • 10 = \$850, the minimum amount. If all 170 children wore a homemade outfit, he would give out 85 • 20 = \$1700. The difference between these amounts is \$850.

28. Since $243 = 3^5$, which is not divisible by 4, there must have only been 3 women, consisting of a woman, her mother, and her grandmother.

29. The store owner lost \$0.90 on the two transactions. To find the cost of the costume on which he made a 25% profit, solve the equation $1.25x = \$18.00$ where x represents the storeowner's cost. To find the cost of the costume on which he took a 20% loss, solve the equation $0.80y = \$18.00$. We find that $x = \$14.40$ and $y = \$22.50$. Therefore, his total cost was $x + y = \$36.90$, but he sold the costumes for \$36. The difference is \$0.90.

30. 3. Let s be the number of small pumpkins, m the number of medium-sized pumpkins, and l the number of large pumpkins. We know that $2l + 1m = 8s$ and that $1l = 1m + 1s$. Solving the first equation for m results in $m = 8s - 2l$, and substituting this result into the second equation gives us

$$1l = 8s - 2l + 1s \rightarrow 3l = 9s \rightarrow l = 3s.$$

31. The equation is

$$y = \frac{8a^3}{x^2 + 4a^2},$$

and the graph of the curve is called the witch of Agnesi, named for Maria Agnesi. The curve can be constructed by placing a circle tangent to a pair of parallel lines $2a$ units apart. Through one of the points of tangency—for example, point O—draw a line that intersects the circle at point P and draw another that intersects the circle at point Q. Create the right triangle with $\overline{PQ}$ as the hypotenuse and with one side parallel to the two original parallel lines. The trace of the vertex of the right angle creates the curve called the witch of Agnesi.

To eliminate the parameters, solve each equation for θ, obtaining $\theta = \tan^{-1}(x/2a)$ and $\theta = \cos^{-1}(\sqrt{x/2a})$. Set these two expressions equal to each other and take the sine of each side, as follows:

$$\sin\left(\cos^{-1}\left(\frac{\sqrt{y}}{\sqrt{2a}}\right)\right) = \sin\left(\tan^{-1}\left(\frac{x}{2a}\right)\right)$$

$$\frac{x}{\sqrt{x^2 + 4a^2}} = \frac{\sqrt{2a - y}}{\sqrt{2a}}$$

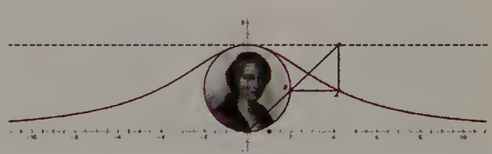
$$\frac{x^2}{x^2 + 4a^2} = \frac{2a - y}{2a}$$

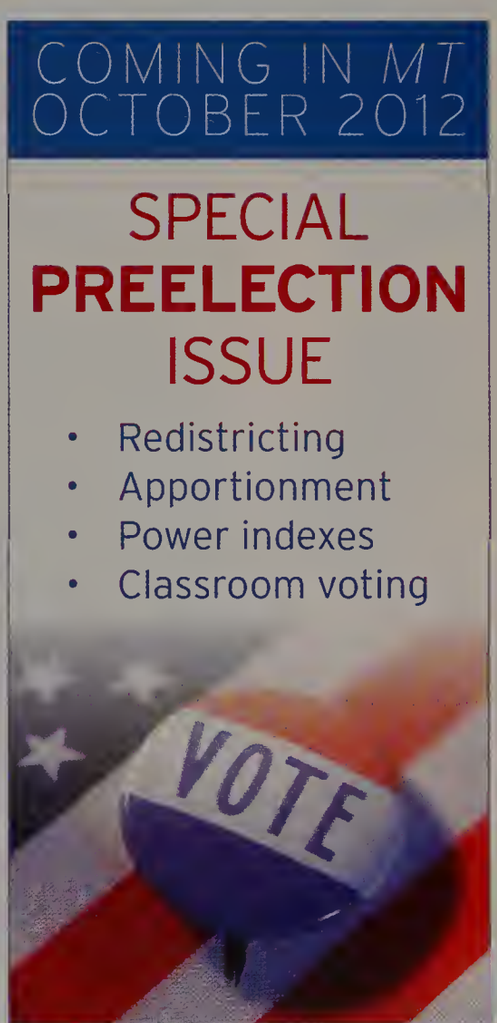
$$\frac{2ax^2}{x^2 + 4a^2} = 2a - y$$

$$y = 2a - \frac{2ax^2}{x^2 + 4a^2}$$

$$= \frac{2ax^2 + 8a^3 - 2ax^2}{x^2 + 4a^2}$$

$$= \frac{8a^3}{x^2 + 4a^2}$$





Fielding an After-School Mathematics Lab

Teachers in an urban high school design a learning environment for at-risk mathematics students.

Christina M. Punches-Guntsch and Erin N. Kenney

Many students will need remedial work in mathematics during their high school years. Some studies note that roughly 25 percent of students entering the ninth grade will need mathematics remediation (Balfanz, McPartland, and Shaw 2002). Some sort of help will be needed to fulfill NCTM's (2000) vision of a mathematics classroom that involves students having access to mathematically rich problems and being engaged in solving them.

The high school where we taught is located in a large urban school district in Kentucky. The school's enrollment is 958 students, of whom 70 percent qualify for free or reduced-cost lunches. For the past three years, we have been involved in the development of an after-school remediation and enrichment program for high school students that we call the Mathematics Lab.

The conception for the Mathematics Lab—five one-and-a-half hour after-school sessions that run from October through March—grew out of an idea that the school's mathematics resource teacher had. It has since become a collaborative program; teachers work together to design and implement the program and to recruit students. The program's goals are to intervene with students who are struggling to pass state mathematics assessments and to build a community of students excited about mathematics.

For the program to be successful, students must be excited about participating in the labs and spending time learning after school. We will describe the Mathematics Lab's structure, the implementation process, and how student achievement data are used to inform development.

THE PRESEASON

Several activities and steps are involved in organizing the Mathematics Lab. First, a team of mathematics teachers interested in working in the labs must meet to design and organize the year's activities. Next, they identify the dates when the labs will be held. This step involves working around both academic and extracurricular schedules. Once the dates for the labs are determined, students are recruited, and activities are designed.

The Mathematics Lab teachers target eleventh-grade students. The state of Kentucky's mathematics test is given during junior year, and those receiving a novice score (students who rarely demonstrate mathematics skills) are required by the state to participate in twelve additional recovery hours during their senior year. Students are identified as potential Mathematics Lab participants by recommendations from their previous teachers on the basis of their performance in class and their

grades in mathematics classes. Once students are identified, the school sends a letter to parents to inform them about the Mathematics Lab experience.

Students are also invited to an assembly promoting the Mathematics Lab experience. When recruiting juniors to participate, seniors who attended the Mathematics Lab the previous year are brought back to give testimony about the program's benefits. Although the idea is to reach out and encourage the identified students, all students who are interested in attending the program are welcome. Incorporating all students gives a different dynamic to the labs and builds a community of students who can come together to enjoy mathematics.

Students must *willingly* attend the after-school mathematics lab sessions. The rewards for students vary according to the number of labs that each attends. There are many different approaches for reaching students, but finding the right rewards is important for enticing students into the labs. Rewards have included providing a snack at each mathematics lab as well as a mathematics T-shirt at the completion of all five labs. The most popular incentives have been the opportunity for participants to opt out of their final mathematics exam or go on an all-day mathematics field trip to the local zoo. The Mathematics Lab teachers create a math trail that requires students to work through mathematics concepts as they explore the zoo. Funding for the snacks, T-shirts, and the zoo trip are provided through the school's Title I funding, which includes a student recognition fund and a free and reduced-cost lunch fund.

An important step in the preparation phase for the Mathematics Lab is the selection of seniors to work with teachers; these students are referred to as mathematics coaches. The students accepted will work as aides during the individual labs and encourage juniors to attend.

Once the dates for the labs have been set and the target audience has been determined, the teachers develop the content for the five labs. Taylor (2004) believes that the process of teachers researching student lessons and planning together as a group will help reach students. First, the teachers determine the topics for the five labs. The team of mathematics teachers collaboratively reviews data from previous state assessments and discusses weaknesses exhibited by students within their first- and second-year algebra and geometry classes. Pinpointing student weaknesses helps the teachers identify topics to research for the labs. In our school district, the adoption of an advanced-level curriculum has left some students at the high school level behind. For the labs, it is critical to choose activities that strengthen students' intermediate and advanced reasoning skills (Balfanz et al. 2002).

The goal is for students to come to the labs not only to work on mathematics but also to be actively engaged. To reach students through remediation, we must give them tasks that complement their classroom experience. Tasks that students view as beneath them or tasks that they feel they have already completed in class will undermine the process of advancing their mathematics career (Balfanz et al. 2002).

Each of the five labs has a lead teacher, who will then identify potential labs for the given topic. The lead teachers use a curriculum that requires students to discover and investigate mathematics problems on their own. The Internet and curriculum and professional development conferences are used as sources to find labs that are student centered, interactive, and hands-on. After the lead teachers have researched their labs, they reconvene to share and discuss potential content for the labs. In a lesson study–like environment, each lead teacher presents his or her potential lab to the other teachers and the student mathematics coaches as if they were the participating students (Grant and Kline 2004). This approach allows all the teachers to discuss the strengths and weaknesses of each lab and ways to improve it. This process is an important step in the development of successful labs.

A SAMPLE ACTIVITY: ALGEBACHEE BALL

One example of a lab chosen for the students to experience is called Algebachee ball 1999–2009). Algebachee ball, adapted from the mathematics lab website (www.themathematicslab.com), was chosen because it involved curricular areas that included the ability to work through problems using number sense, order of operations, properties of exponent, operations with variables, and evaluating an expression. According to state test data, all these areas needed improvement. Algebachee ball helped address the areas needing development. In addition, when we met to plan the labs, we noted that many students lacked number sense. Having additional opportunities to actively practice these skills would be helpful.

$2x$	$-x$	25	x^2	$-3x$
50	$-x^2$	$3x$	-25	100
$2x^2$	-100	$-3x^2$	-50	x

Fig. 1 The Algebachee board should be projected so that all participants can see it.

The setup of Algebachee ball involves the Algebachee board (see **fig. 1**), two dice, five paper airplanes (or wads of paper), and an Algebachee scorecard (see **fig. 2**). Algebachee ball can be played by an individual or in teams, depending on students' ability level. Seven students volunteer to start the game. One student will roll a die to determine the sign of x; even outcomes indicate a positive x, and odd outcomes indicate a negative x. Another student rolls a die to determine the value of x. The Algebachee board needs to be shown on an overhead projector for the next five students to complete their task—throwing the paper airplanes (or wads of paper) at the board to select the terms.

Once five terms have been selected, students will use them once and only once, along with the operations (addition, subtraction, multiplication, division) and squaring and parenthesis to create an expression that will produce a maximum value. The student who creates the largest value is the winner. The game's difficulty can be enhanced by not using calculators. Once students become proficient at discovering the largest number possible, they are challenged to find the smallest positive number.

This activity can be introduced in numerous ways. Working through the game before Mathematics Lab gave the instructors a chance to develop a strategy for presenting the game to students. Finding the smallest positive number was a tweak added after collaborative teacher discussion. Teachers also discussed the pitfalls of using the activity, such as students improperly entering their expressions into the calculator, leading to a discussion on the need for students to demonstrate their work. Another key component of the activity is to have instructors circulate around the room and verify correct answers before students present their work to the group.

Demonstrating our strategies to the group as teachers encouraged discussion about the effects of operations on the final outcome. From this we learned that it is important to create similar discussions in the classroom. For teachers, the challenge of beating one another for the highest score enhanced the mathematics processes behind the game. Creating a competitive and challenging classroom for the students increases the activity's appeal.

<table>
<tr><th>Round</th><th>Sign of x</th><th>Value of x</th><th colspan="6">Algebachee Scorecard</th><th>Total Score</th></tr>
<tr><td>1</td><td></td><td></td><td>Term</td><td>Term</td><td>Term</td><td>Term</td><td>Term</td><td></td><td rowspan="5"></td></tr>
<tr><td colspan="3">Must use at least once:</td><td>+</td><td>−</td><td>×</td><td>^2</td><td>/</td><td>( )</td></tr>
<tr><td colspan="8">Expression 1:</td></tr>
<tr><td colspan="8">Expression 2:</td></tr>
<tr><td colspan="8">Evaluate:</td></tr>
</table>

Fig. 2 This form can be used as an Algebachee scorecard.

Following Directions in the Mathematics Lab

- You must follow teacher directives at all times. This includes during the mathematics activity and during the transition.
- You must be an active participant at all times. This means that you will work on mathematics in the mathematics lab; you will not work on homework or have other materials out.
- You will work in groups to do the labs. You must be an active participant in your group.
- If you fail to follow directions, you will sit out the activity, you will not receive full credit, and you will not be invited back to Mathematics Lab.

Punctuality

- *You may not come late or leave early.*
- You must be in the large cafeteria by 2:30. If you are late, you will not receive full credit.
- If you leave early, you will not receive full credit for your participation. (Doctor or dental appointments are not excuses to leave early.)

Cell phone use

- You may not have your cell phone out at any time, just as during the regular school day.

Head phones/iPods®/MP3 players

- These should be put away in your locker, backpack, or pocket. They should not be seen or heard.

Restroom

- Use the restroom before your Mathematics Lab session begins.
- If you need to use the restroom during the lab session, ask for and obtain the teacher's permission and return promptly.

Fig. 3 Expectations for participating in the Mathematics Lab are spelled out clearly.

Name: _____ Lab name: _____

What mathematics concepts did you learn today?

How will this lab session prepare you for the state assessment?

Give some suggestions for improving this particular lab session or the Mathematics Lab program in general?

Rate your overall experience today from a range of 5 to 1, with 5 indicating "very satisfied" down to 1 indicating "very dissatisfied":

5 4 3 2 1

Fig. 4 At the end of each lab, students fill out an exit slip such as this.

THE MATHEMATICS LAB SEASON

Students are assigned to five different randomly chosen groups that rotate together throughout the entire mathematics lab season. The lead mathematics teacher who initiated the design of the lab, other supporting mathematics teachers, and student mathematics coaches will stay with the same lab for the entire season. In their first lab, students are given a list of expectations (see **fig. 3**). Although the Mathematics Lab is an after-school program, students are still expected to behave as they would in a classroom and to participate to earn credit for attending the lab; good behavior counts toward their incentives. Each student receives a Mathematics Lab folder to collect the information from all the labs. Students also receive an exit slip (see **fig. 4**) at the end of each lab; they use this to rate the experience and describe their learning goals. The exit slips are also used as a record of student attendance.

THE POSTSEASON

At the end of the school year, teachers analyze the results of the lab and reward students. The analysis process includes recording the total number of students who attended four labs and the number who attended all five labs. When the results of the state assessment arrive, teachers meet to determine the scoring ranges of students who attended the labs and the scoring ranges of those students who were predicted to struggle on the assessments. These data are helpful in determining the success of the labs in general and for specific topics addressed in the labs.

Table 1 summarizes the data for eleventh-grade students during the three years in which we have offered the Mathematics Lab. The data show that students have made some strides in their performance. The state of Kentucky evaluates student performance on the state achievement test as *novice*, *apprentice*, *proficient*, or *distinguished*. The first row shows the decrease in the percentage of students who received a score of novice on the state assessment. During the three years in which we have been offering the Mathematics Lab, the percentages of proficient and distinguished students have remained steady across the three years. In addition, the number of students attending the Mathematics Lab has increased each year. In the 2008–2009 school year, we began specifically tracking the performance of the juniors so identified as potential novice and their performance on the state assessment.

CONCLUSION

The Mathematics Lab offers extra help to all high school mathematics students while targeting those who struggle the most. Using a combination of appropriate tasks and teacher collaboration to develop and

Table 1 Mathematics Lab Data Table

	School Year			
Student Performance Level	**2005–2006 (Prior to Lab)**	**2006–2007 (Lab Began)**	**2007–2008**	**2008–2009**
1. Novice on state assessment	44%	37%	36%	18%
2. Proficient or distinguished	23%	23%	23%	33%
3. Number of juniors who attended four or more labs	NA	32	32	79
4. Number of juniors who attended all five labs	NA	0	6	56
5. Percentage of juniors who attended the lab and scored above novice	NA	70%	63%	82%
6. Number of juniors identified as potential novices	NA	NA	NA	44
7. Number of identified juniors who attended all five labs	NA	NA	NA	20
8. Number of juniors who scored above novice and attended all five labs	NA	NA	NA	9

fine-tune these tasks, our high school has found success in offering this after-school program.

The goal is to have all students achieve at a high level, but to reach this goal we have to change how we approach teaching and learning. The collaborative approach we have used has created an environment in which teachers are more willing to work together and share material. This sense of shared vision has enhanced the methods used in individual classrooms through the new ideas and approaches gained through fellow teachers and student mathematics coaches.

The strategies used in the Mathematics Lab have built a community of students who enjoy mathematics and are willing to help other students.

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EMPIRICAL APPROACHES TO THE

Experiments with random numbers give a new twist to this familiar problem.

Alfinio Flores and Kevin M. Cauto

Developing a sense of the distribution of random outcomes requires at least two things. First, students need a large number of experiences with the same probabilistic situation, a rare event in most classrooms. Second, they need to encounter a variety of probabilistic situations, including equally likely events (such as tossing a coin or one die) and unequally likely events (such as rolling various sums with a pair of dice). Often students form their conceptions about probability from a limited number of experiences and are frequently not aware of their misconceptions and poor intuition. These poorly formed notions must be consciously identified so that students can modify them.

One way to do so is to have students make and justify predictions about a probabilistic situation in which the results will be surprising and unexpected. This approach allows any misconceptions to become explicit. When students articulate their ideas before they experiment or analyze the situation, they maximize the potential that such situations will make them rethink their basic assumptions (NCTM 1989, p. 110).

One approach to developing understanding of basic concepts of probability is to use simulations to construct empirical probability distributions (NCTM 2000, p. 324). The use of technology can facilitate students' learning of probability in at least two ways. First, students can generate a large number of simulations in a short time; thus, they can observe variability from one experiment to the next. Second, because samples are generated fairly easily by computer or calculator, students can focus

their attention on analyzing the data (NCTM 2000, p. 254). Students can thus perform experiments to develop probabilistic intuition and concepts before dealing with theoretical probabilities. Analyzing the theoretical probability in light of the empirical results allows students to see the relationship between the two.

UNEXPECTED: AT LEAST ONE REPEATED BIRTHDAY IN A GROUP

Students are surprised to find that two students in the same class will have the same birthday. Ask students to state the probability that at least two people in a group of 40 will have the same birthday (the same month and day, not necessarily the same year). Many will think that the probability is fairly low, perhaps because 40 is a small number compared with 365. Although students see fairly easily that to be certain that two people have the same birthday the group must have 366 persons (we assume that a year has 365 days), they have not developed an appropriate intuition about the likelihood of having at least one repeated birthday in a group of a certain size.

We will describe two activities in which students conduct experiments with random numbers so they can see that having at least one repeated birthday in a group of 40 is not that unusual. The first empirical approach was conducted by author Cauto in a secondary school methods course. The second empirical approach was used by author Flores with in-service teachers. In both, the participants pretended to be secondary school students. In a third activity, students use a calculator program to determine the theoretical probability.

To simplify the experiment, we will make several assumptions. We will use a year with 365 days, thus disregarding birthdays on February 29. We will assume that each day is equally likely for birthdays. In real life, this assumption is not quite true; in the United States, the daily average of births is slightly higher during the summer months (July–September) and slightly lower in January (James 2005). Rather than listing the birthdays by month and day, we will use the numbers from 1 to 365, inclusive. Thus, 2 corresponds to January 2, 32 corresponds to February 1, and 365 corresponds to December 31.

THE FIRST EMPIRICAL APPROACH

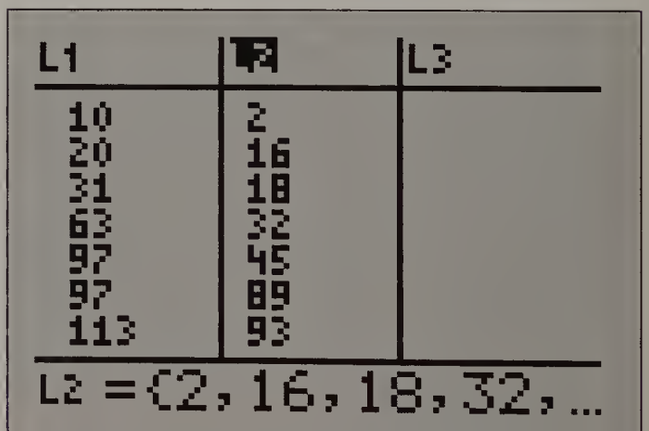
Before beginning this activity, let students guess the probability that in a group of 23 people there is at least one repeated birthday. Write students' guesses on the board. To estimate this probability, students will individually perform an experiment and, on the basis of the class results, will compute the empirical probability.

Students can use a computer or a graphing calculator to generate lists of random numbers. For example, when using the TI-84 Plus, students press the **MATH** key and move the cursor to the **PRB** (probability) menu to show the option **randInt(**. For this function, they can choose the range of random numbers and the number of these they wish to generate. For example, **randInt(1,365,23)** will generate 23 random whole numbers from 1 to 365, inclusive. We can instruct the calculator to store the numbers in a given list by using the **STO▸** key. So if we type **randInt(1,365,23)STO→L2**, the calculator will generate 23 random numbers and will store them in **L2**.

Once the data are stored in a given list, students can order them. To do so, students press the **STAT** key; then, from the **EDIT** menu, they choose **SortA(** and enter the list they want to sort—for example, **SortA(L2)**. In **figure 1**, two lists have been generated. After sorting, we see that **L1** has one entry repeated: 97. The repeated number means that two people had the same birthday (97 corresponds to April 7).

Students can generate, store, and sort their own lists. As students review their lists, they can see whether there are repeated numbers or not. When we conducted this activity with a group of 20 students, each of them generated a list of 23 numbers, representing the birthdays in a group. In 11 of the 20 cases, there was at least one repeated birthday in the group. The participants were quite surprised that in so many simulations (55%) there was at least one repeated birthday.

Increase the size of the groups—say, to 40—and let students again guess the probability that in a



Simulation 1: {10, 20, 31, 63, **97, 97**, 113, 122, 136, 152, 169, 179, 192, 212, 213, 222, 228, 294, 332, 342, 354, 360, 363} (repeated birthdays are highlighted)
Simulation 2: {2, 16, 18, 32, 45, 89, 93, 94, 99, 103, 111, 112, 117, 150, 220, 245, 283, 290, 299, 309, 310, 320, 350}

Fig. 1 Two lists of ordered birthdays can be generated as shown.

group of 40 people there is at least one repeated birthday. Write these guesses on the board. To simulate the situation with the calculator, students would need to modify slightly the calculator instruction to randInt(1,365,40)STO→L1, sort the list, and then verify whether there are any repeated numbers. The results for the whole class will again serve to estimate the empirical probability.

When students generate longer lists, it becomes obvious that the probability of at least one repeated birthday for the given size of the group is fairly high. For example, when 20 students generate lists of birthdays for groups of 40 people, they find that in the vast majority of the lists there is at least one repeated birthday. In one instance, 20 simulations of groups of 40 people yielded 18 cases of at least one repeated birthday and only 2 cases with no repeated birthdays. The empirical probability is thus 90 percent.

Before students do the experiments, very few guess that for groups of only 23 students the probability is already about 50 percent that two people will have the same birthday or that the probability of at least one repeated birthday in a group of 40 is almost 90 percent. For most students, 23 is too small a number compared with 365 to think that the probability is about 50 percent. Because the results are so surprising, it is important to solidify students' understanding of the probabilistic situation through another simulation.

A SECOND EMPIRICAL APPROACH

In the second activity, students run a program that simulates adding people at random to a group until there is a repeated birthday. For a given group, the program randomly adds a new birthday to the group and records it until there is a repeated birthday. The calculator will display how many people were in the group when a birthday was first repeated and does so for 50 groups (see **fig. 2**).

Table 1 shows the results of running the program. We see that in these 50 results there were never more than 45 people before a repeated birthday occurred. Of course, running the program again will give slightly different results. Students can determine the median of the data in **table 1** (22). This is another way to see that the probability of repeated birthdays for groups of more than 23 people is greater than 50 percent.

The graph shown in **figure 3** represents the results of performing the simulation 200 times. Students can see that in 107 cases (more than 50%), the groups had 24 people or fewer when the repeated birthday occurred.

PROGRAM:CUMPLE1

For(N,1,50,1)	Start of loop for 50 groups
365→dim(L4)	List for 365 birthdays
Fill(0,L4)	Zero for each day
0→P	Zero people in the group
Repeat max(L4)>1	Loop group; instructions executed until one birthday is repeated
(1+int(365*rand))→R	New birthday added at random
(1+L4(R))→L4 (R)	Birthday tally incremented
1+P→P	Number of people incremented
End	End of loop for group
Disp P	Number of people in group displayed
Pause	Pause between groups (press ENTER to continue)
End	End of loop for 50 groups

Fig. 2 The calculator commands are on the left; the explanation of each line is on the right.

Table 1 Frequencies for Groups of Different Sizes

Number of People in Group	Frequency	Number of People in Group	Frequency	Number of People in Group	Frequency
3	1	16	1	30	2
4	2	18	1	31	1
5	1	19	3	32	1
7	1	21	2	34	1
8	2	22	4	35	3
9	2	23	3	36	1
11	2	24	1	39	1
12	1	25	1	42	1
13	2	27	1	43	2
14	2	28	3	45	1

THEORETICAL PROBABILITY

After students have done several simulations to see the experimental probability for having at least one repeated birthday in a group, they can deal with the theoretical probability of the situation. To find the probability of at least one repeated birthday in a group, students may find it easier to think first about the probability that there are no repeated birthdays among a group of people. That is, they will compute the probability of the complementary event first.

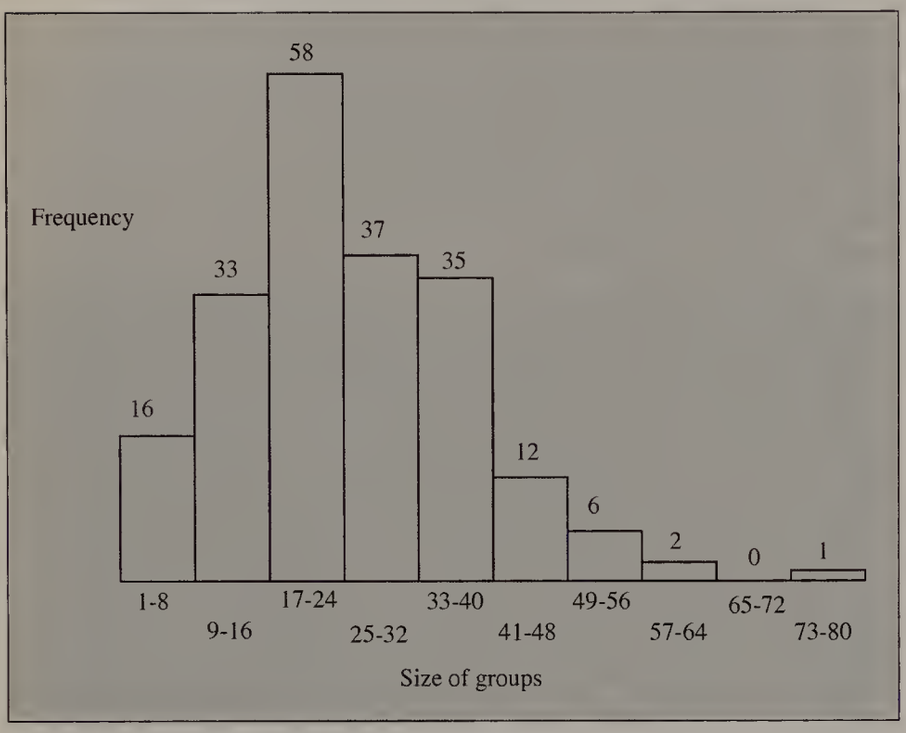


Fig. 3 The histogram displays data on the size of groups for 200 simulations.

PROGRAM:BIRTHDAY

1→Q	Probability of no repeated birthdays for one person
FOR(N,1,100,1)	Beginning of loop
Q*(365-N)/365→Q	Probability of no repeated birthdays adding one more person
DISP N+1, 1-Q, " "	Number of people plus probability of repeated birthday
Pause	Program pauses (hit ENTER to continue)
End	End of loop

Fig. 4 This program displays the number of people and the probability of the complementary event of at least one repeated birthday.

If the group consists of only one person, there will be no repeated birthdays, so the probability is 1. If the group consists of two people, the probability of not having a repeated birthday is 364/365 because the birthday of the second person can be any of the 364 days other than the birthday of the first person. If the group consists of three people, we multiply the probability that the first two people have different birthdays by the probability that the birthday of the third person is one of the other 363 days. We compute

$$\frac{364}{365} \cdot \frac{363}{365}.$$

If the group consists of four people, we need to multiply the previous result by

$$\frac{362}{365},$$

which is the probability that the new birthday is different from those already in the group.

Thus, we can compute the probability of different birthdays in a recursive way. If we know the probability for a group of n people, we can compute the probability for $n + 1$ people by multiplying the product corresponding to n people by

$$\frac{365 - n}{365}.$$

If Q is the probability that in a group of $n + 1$ people there are no repeated birthdays, then

$$Q = \frac{364}{365} \cdot \frac{363}{365} \cdot \cdots \cdot \frac{365 - n}{365}.$$

Therefore, the probability of having at least two people with the same birthday is $1 - Q$.

Students can run a short calculator program to determine the probability of repeated birthdays for successive numbers of people in a group. **Figure 4** shows a program for the TI-84. For each one of 100 people added in succession, the program computes the probability that the new person's birthday is

Table 2 Theoretical Probabilities for Repeated Birthdays

Number of People in Group	Probability of Repeated Birthdays	Number of People in Group	Probability of Repeated Birthdays
2	.002739726	41	.903151612
4	.016355913	50	.970373580
23	.507297234	57	.990122459
30	.706316243	70	.999159576

not the same as any of the people already accounted for and multiplies it by the probability that there are no repeated birthdays among people already in the group.

Students can run the program and see what the probabilities for successive numbers of people are. **Table 2** displays the results for a few numbers: 2, 4, 23, 30, 41, 50, 57, and 70. Students will see that for 70 people, the probability of repeated birthdays is more than 99.9 percent, even though 70 may seem a relatively small number compared with 365.

CONCLUSION

A common misconception about random samples is that outcomes are "spread out" more or less evenly among possible results. Thinking about 23 people compared with 365 days, students may think that there are a lot of slots available. They may also think that because, on average, birthdays are spread out two per month, it is very unlikely that two people will have the same birthday.

However, students need to develop an understanding that sometimes results within a given random sample are clustered or repeated. Students must also understand that the distribution within a particular sample is not always uniformly or symmetrically distributed among the possible results. Performing simulations like the ones described here can help students develop such understanding.

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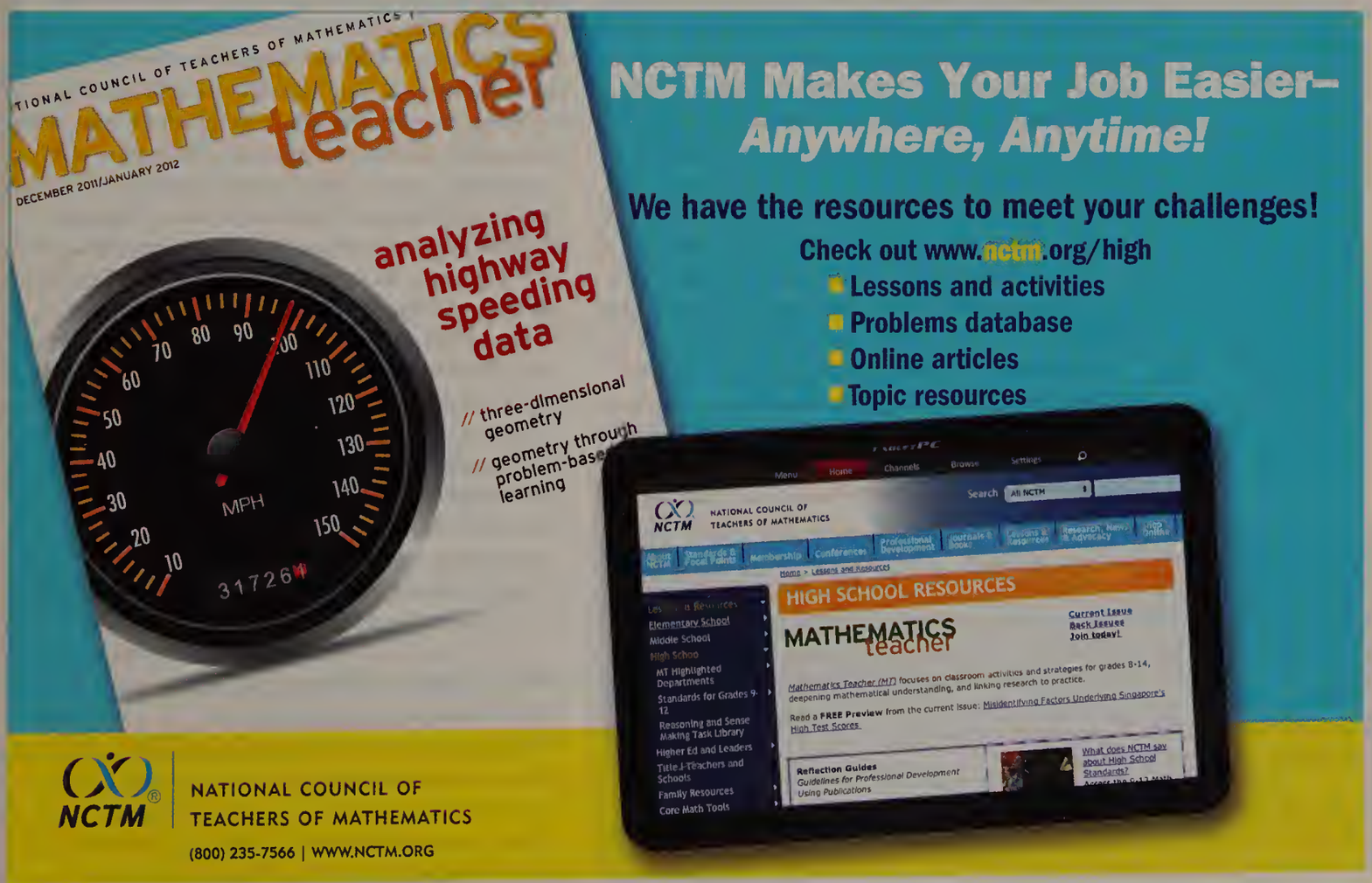

Thomas R. Post, Debra S. Monson, Edwin Andersen, and Michael R. Harwell

Integrated Curricula and Preparation for College Mathematics

In the early 1990s, after a long series of disappointing results on national and international mathematics achievement tests—for example, TIMSS (1998) and NAEP (Campbell, Hombo, and Mazzeo 2000)—the National Science Foundation (NSF) funded the development of thirteen complete mathematics programs at the elementary school, middle school, and secondary school levels.

By NSF edict, each reform curriculum had to align with the basic process principles and content specifications articulated in NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989). At the secondary school level, these NSF-funded curricula, released between 1997 and 1999, extended the mathematical domains addressed each year by including attention to algebra, geometry, probability, statistics, and topics in discrete mathematics. The pedagogical guidelines were also more expansive and quite different from the methodology commonly used at the time. As a result, implementing these materials posed a different set of challenges for teachers.

Many teachers were quite happy with the new curricula and impressed with the levels of student involvement and accomplishment. However, some parents and mathematicians were opposed to them, referring to the new curricula as "fuzzy mathematics." In response, in 2007 NSF funded the Minnesota Mathematics Achievement Project (MNMAP) to explore the empirical relationship between high school mathematics curricula and college-level mathematics performance.

MNMAP is a longitudinal program of research. Various subsets of its database of more than 16,000 students from 366 high schools who attended thirty-two four-year regional colleges and universities in the upper Midwest were involved in the MNMAP studies. All students graduated from a Minnesota high school and completed three or more years of college-intending high school mathematics. This group presented a unique opportunity to examine the mathematics achievement of students who had recently completed one of two distinctively different types of high school curricula—one NSF-funded and one commercially developed. These students have now graduated from regional four-year colleges and universities.

We will summarize five of these MNMAP studies, each of which used quantitative methodology. All studies were concerned with the efficacy of three first-edition secondary school curricula—Contemporary Mathematics in Context (Core Plus), Interactive Mathematics Project (IMP), and Mathematics Modeling Our World (MMOW)—as measured by student performance in college mathematics. Each program received an additional grant from NSF to prepare second-edition materials.

TWO VIEWS OF HIGH SCHOOL MATHEMATICS CURRICULA

The three NSF-funded curricula considered here offer students opportunities to communicate and reason mathematically in ways not found in traditional textbooks, referred to here as commercially developed curricula. In the NSF-funded curricula,

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problems are introduced within some mathematically rich, real-world context. Rarely are raw numbers or symbols manipulated simply for practice.

Two criticisms of NSF-funded curricula were that they did not pay enough attention to algorithmic development and that more extensive practice needed to be included in the lessons. The cognitive perspective underlying NSF-funded curricula expands the nature and extent of mathematical and instructional goals. This means that algorithmic development and maintenance is one of several important goals but, because the content has been expanded, a lesser portion of classroom time is available for the development and maintenance of skills and procedures. Critics believed that this approach would leave students unprepared for college-level mathematics. This hypothesis is unsupported by empirical evidence.

Method

In the past, the school mathematics curriculum has not received the research attention that it deserves. Rarely have two curricula been contrasted to see which has been more effective over the long run. The NSF-funded curricula are probably the most researched curricula in our nation's history. Previous studies assessed topic-specific achievement, calculus 1, problem-solving ability, attitude, and tenacity, but we identified no studies that used large databases from multiple postsecondary institutions to follow students through four full years of college coursework. The MNMAP studies discussed here are an attempt to fill this void and provide insight into the relationship between high school mathematics curricula and college-level performance.

In educational settings, units of analysis—students, schools, institutions—are often embedded, or nested, within larger units. For example, students are nested within classrooms that are nested within schools. Units within a larger unit tend to be more similar to one another and are therefore not independent (e.g., students in a classroom tend to be more similar to one another than they are to students in a different classroom). Traditional analyses, such as multiple regression, does not take this dependence into account, but hierarchical linear modeling, the primary analytic tool used in the MNMAP studies, does, thus significantly improving our ability to understand relationships among the many meaningful and relevant variables that typically are present in educational studies.

The first four MNMAP studies discussed here compared the achievement and course-taking patterns of students who had completed three, four, or five years of an NSF-funded curriculum with students who had used a commercially developed program or the University of Chicago School Mathematics Project. The fifth study relates to the likelihood of students completing a college STEM major—engineering or mathematics, in particular—as a function of the type of their high school mathematics curriculum.

New college students in 2002 and 2003 were some of the first students to have completed three or four years of one of the NSF-funded curricula. Some also completed a fifth year of secondary school mathematics (usually calculus), neither a non-NSF-funded nor a noncommercially developed course. Students in the latter cohort used materials published by one of approximately twenty programs, including those of Laidlaw, Holt, Addison Wesley, and Saxon. Students in the NSF-funded cohort were enrolled in courses using either Core Plus, IMP, or MMOW. The MNMAP website (www.cehd.umn.edu/ci/mnmap/) contains additional information about each study discussed here. In the three longitudinal studies, MNMAP examined all students' mathematics coursework taken over eight semesters of college work.

The MNMAP studies discussed here are an attempt to provide insight into the relationship between high school mathematics curricula and college-level performance.

Independent Variables

Student level variables (independent variables or covariates) were (1) high school curricula compared — Variable of major interest in each of these studies: (1a) NSF-funded vs. commercially developed curricula; (1b) Core Plus vs. commercially developed curricula, (1c) NSF-funded vs. commercially developed and UCSMP; (2) ACT scores; (3) number of years of high school mathematics (three, four, or five); (4) high school mathematics GPA; (5) high school percentile rank; (6) college major; (7) gender; (8) ethnicity; and (9) basic skills state test scores.

Dependent Variables

The dependent variables considered included these: (a) the difficulty level of the first college mathematics course taken (developmental, college algebra, calculus, etc.); (b) the difficulty level of the last college mathematics course taken (developmental through upper-division mathematics courses); (c) the grades or patterns of grades that students earned in mathematics courses (achievement); (d) the number of mathematics courses completed (persistence); (e) the pattern of difficulty levels of all mathematics courses completed over multiple semesters; (f) the likelihood of completing a STEM

Table 1 Summary of the Design and Participants of the Five MNMAP Studies

Study	Reference	Participants	Independent (I) and Dependent (D) Variables	Design
1	Harwell, Post, et al. (2009)	$N = 1{,}667$; 85 high schools	I: (1a), (2), (3), (4), (5), (8), (9) D: (a), (c)	Retrospective quasi-experimental, cross-sectional student data from a single institution at one point in time
2	Harwell, Medhanie, et al. (2011)	$N = 1{,}588$; 209 high schools	I: (1b), (2), (3), (4), (6), (7), (8) D: (a), (b), (c), (e), (g)	Retrospective, longitudinal analysis (four years of college coursework); data from a single institution
3	Post et al. (2010)	$N = 4{,}144$; 266 high schools	I: (1c), (2), (3), (4), (6), (7), (8) D: (a), (b), (c), (d), (e)	Retrospective, longitudinal analysis; data from a single institution
4	Harwell, Post, et al. (in press)	$N = 10{,}190$; 266 high schools; 32 four-year postsecondary institutions	I: (1c), (2), (3), (4), (6), (7), (8) D: (a), (b), (c), (e), (g)	Retrospective, longitudinal analysis; data from multiple institutions
5	LeBeau et al. (2012)	$N = 3{,}459$; 1,166 STEM majors; 537 completed an engineering or mathematics degree	I: (1c), (2), (3), (5), (7), (8) D: (f), (h)	Retrospective, cross-sectional analysis; data from a single institution

major in college; (g) the student's grade in a first calculus course; and (h) within STEM majors, the likelihood of completing an engineering or a mathematics major. This research was conducted initially with students from one large university and later with students from thirty-two four-year regional institutions varying in size, selectivity, mission, and type (public or private). The key findings are consistent over the five studies and across the thirty-two institutions.

See **table 1** for a summary of the design and participants for each study. Note that the references to independent and dependent variables refer to the lists above. **Table 2** provides a summary of the results across the first four studies.

SUMMARY OF RESULTS

Study 1: Harwell, Post, et al. (2009)

When we examined the entire NSF-funded cohort, we found differences in the difficulty level of initial course enrollment favoring students in commercially developed programs. These enrollments, from both types of curricula, included developmental (non-credit-bearing) and nondevelopmental courses (college algebra, calculus 1 and 2, and higher-level courses). There were no curricular-related differences in the grades attained in these courses.

Study 2: Harwell, Medhanie, et al. (2011)

When we examined only students who took two or more college mathematics courses, the curricular-related differences mentioned in study 1 disappeared. That is, there were no differences in the difficulty level or achievement (grades) of the initial course enrollment. Further, there were no differences in the longitudinal difficulty levels of coursework completed or in the grades attained in those courses examined over eight semesters of coursework. We also found no curricular-related differences in the number of mathematics courses completed.

Study 3: Post et al. (2010)

When we partitioned students according to their previous achievement scores (forming three groups), the middle group (those with ACT scores of 22 to 26) and the high group (those with ACT scores higher than 26) showed no significant differences in any of the dependent variables (the difficulty levels of the initial and final courses, achievement, and persistence) that we examined. For the group of students with the lowest ACT scores, we found a difference in the difficulty level of initial course enrollment (primarily developmental) favoring the cohort using commercially developed materials, with no curricular differences in grades earned.

Table 2 MNMAP Summary of Results for Studies 1–4

	Number of Students	Difficulty Level of Initial College Mathematics Course	Grade of Initial College Mathematics Course	Persistence (Number of College Mathematics Courses)	Longitudinal Grade Trajectories	Longitudinal Difficulty Trajectories
Study 1	1,667	CD > NSFF	NSD			
Study 2	1,704	NSD	NSD	NSD	NSD	NSD
Study 3 (ACT strata)	4,144	Low 1/3: CD > NSFF Middle and Top 1/3: NSD	NSD	NSD	NSD	NSD
Study 4 (Multi-institutional)	10,300	CD > NSFF	NSD (last course)		NSD	NSD

Note: NSFF = National Science Foundation funded; CD – Commercially developed curricula; NSD = No significant difference

Study 4: Harwell, Post, et al. (in press)

When we expanded the sample to thirty-two four-year regional institutions and more than 10,000 students, the results were consistent. When these students were compared with students in the cohort using commercially developed materials, the only difference was in relation to the difficulty level of the first college mathematics course. That is, students in the NSF-funded cohort, regardless of institutional parameters, were as well prepared for college mathematics as their counterparts who used commercially developed materials, suggesting that earlier results related to preparing students equally well for college mathematics are generalizable. Again, students in the NSF-funded cohort enrolled initially in lower-level coursework but achieved equally well in that course and in subsequent courses. We found no differences at any of the difficulty levels of the last college mathematics course completed.

Study 5: LeBeau et al. (2012)

When we restricted our sample to only those students who had completed a STEM major and then to students who had completed an engineering or mathematics major, the results were again consistent. That is, we found no relationship between students' high school curriculum and their likelihood of completing an engineering or a mathematics major in college.

EXPLANATION OF RESULTS

Teachers will notice that we used the terms "equally well prepared," "no differences," and "no curricular relationship" in our summary. What exactly are the implications of these phrases, and why are these implications important for classroom teachers? These results suggest that multiple, very different approaches can prepare high school students equally well for college mathematics.

To qualify for funding, all NSF-funded curricula must have adhered to NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989). It follows that each of these curricula reflect in various ways the "best practices" in our field. Although NCTM's 1989 publication and its *Principles and Standards for School Mathematics* (NCTM 2000) are not exclusively research based, they do reflect many research implications that have accumulated over the past several decades relating to student learning and conceptual development, effective teaching, the sociology of the classroom, uses of technology, and appropriate assessment practices. Classrooms based on these principles portray a very different learning environment—one that is student centered. In such classrooms, the teacher's role changes from primarily that of a lecturer to that of a coach who promotes both student independence and interdependence. This new role routinely highlights the applications of mathematics in the real world and preempts the ever-present student question, "Why do I have to learn this?"

Does such a student-oriented environment become a disadvantage for students later on? Our results suggest that it does not. In fact, these studies suggest that, aside from curricular-related differences in NSF-funded and commercially developed curricula in the difficulty level of the first college mathematics course, primarily for students enrolling in lower-level courses, we found no other differences in any of the dependent variables.

As we attempted to explain these results further, we focused on the characteristics of NSF-funded curricula. Each of the three curricula—Core Plus, IMP, and MMOW—was designed to appeal to a wide spectrum of the high school population. Key elements include the integration of five mathematical domains every year, widespread use of contextually laden problem situations, group-oriented problem solving, and use of a wide variety of

technology, including spreadsheets, graphing calculators, computer algebra systems (CAS), and applications such as The Geometer's Sketchpad® and Fathom Dynamic Data™ software.

These elements engage students in the conceptually oriented aspects of mathematics and its applications while necessarily limiting the time devoted to routine paper-and-pencil manipulations. This approach represents a substantial departure from the status quo and consequently has contributed to the acrimonious interchanges alluded to earlier.

The NSF-funded curricula included in these studies do prepare students for college mathematics equally as well as commercially developed approaches.

The problem-solving aspects of NSF-funded curricula engage students in problems whose solutions often consume several days of class time. This more accurately depicts the interdisciplinary nature of problem solving as it occurs in the real world. The tenacity acquired from these elaborate problem-solving experiences apparently served students well as they pursued college-level mathematics.

These characteristics do not appear only in the NSF-funded curricula. However, they are found there in a more coordinated fashion and in more abundance than in more traditional approaches.

A LOOK AHEAD

The Common Core State Standards for Mathematics (CCSSI 2010), designed to establish national standards for school mathematics, has been endorsed by more than forty states and the District of Columbia. The intent is "the specification of content that all students must study in order to be college and career ready" (CCSSI 2010, p. 4).

CCSSM has developed two major model course pathways—a traditional one and and an integrated one. Specific mathematics content by grade or course level is suggested within each of the specified pathways, including a transpathway perspective of an expanded view of the various mathematical content domains to be addressed each year. The NSF-funded curricular approaches, by virtue of addressing five mathematical domains each year and selecting problems from a wide variety of disciplines, has had an integrated perspective from the beginning.

CCSSM also endorses a series of mathematical practices that relate to NCTM's original five Process Standards: Problem Solving, Reasoning and Proof, Communication, Connections, and Representation. The Common Core Standards for Mathematical Practice are (1) make sense of problems and persevere in solving them; (2) reason abstractly and quantitatively; (3) construct viable arguments and critique the reasoning of others; (4) model with mathematics; (5) use appropriate tools strategically; (6) attend to precision; (7) look for and make use of structure; and (8) look for and express regularity in repeated reasoning (CCSSI 2010, pp. 6–8).

These practices are subtle yet profound, and implementation in the manner intended will represent a substantial departure in both content and method from the status quo. These core practices are not unrelated to more traditional approaches to high school mathematics; rather, these practices are more ably reflected in the NSF-funded approaches. This should not be surprising—these curricula have reflected alternative types of classroom mathematical activities for more than a decade.

SUMMARY AND IMPLICATIONS

Newer approaches to school curricula are always adapted reluctantly, especially so in mathematics. After all, everyone is interested in providing the best possible experiences for his or her own students. In one local suburban district, NSF-funded curricula were implemented for grades K–12 for all students more than a decade ago. Even though this district was high achieving before the change, it is now one of the top two or three highest-performing districts in the state regarding graduation rates, standarized test scores, the number and percentage of students in AP calculus and AP statistics courses, the percentage of students who take four full years of college-intending mathematics (through Core Plus IV), and the percentage of students enrolled in four-year postsecondary institutions (Monson 2011).

Earlier concerns about NSF-funded curricula—primarily that they will not prepare students for college-level mathematics in general and calculus in particular—have not been validated by these studies. The NSF-funded curricula included in these studies do prepare students for college mathematics equally as well as commercially developed approaches. Further, there may also be legitimate precollege pedagogical and sociological reasons for adopting NSF-funded curricula.

For schools and teachers interested in materials that have a decidedly different perspective, that appeal to a wider audience, and that promote a higher degree of student engagement, the NSF-funded curricula are a viable option.

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Nicole R. Juersivich

Investigating the Law of Large Numbers with Visual Basic

By using technology, students can conduct an experiment that quickly simulates a large number of random events. Much research has been done on students' conceptions and reasoning about probability (Jones et al. 2007). Recommendations for teaching probability have included just such use of concrete and digital manipulatives to simulate events as well as students' reflection on their initial predictions and analysis of their experiments and their results (NCTM 2000; Van de Walle et al. 2010). In fact, by using Excel® and Visual Basic to simulate coin flipping, students have been able to capitalize on these technological benefits to investigate, conceptualize, and refine their understanding of the law of large numbers.

THE PROBLEM

Which has the higher probability—
(a) getting exactly 5 heads and 5 tails in 10 flips of a fair coin; or
(b) getting exactly 50 heads and 50 tails in 100 flips of a fair coin?

Informally, the law of large numbers states that the relative frequency of outcomes becomes a closer approximation of the theoretical probability of an event as the size of the data set increases (Van de Walle et al. 2010). Students often misinterpret this law in the context of coin flipping. They believe that, as the number of flips increases, the number of tails and the number of heads approach each other. Therefore, when presented with the problem above, they believe that the correct answer is (b).

AN OVERVIEW

After students make and justify their predictions, they can investigate the problem by using a programmed Excel 2007 workbook containing three worksheets. The first two worksheets (see **fig. 1**) simulate flipping a coin 10 times and 100 times, respectively, and record and display the outcomes in a column graph for each of 1000 trials. Worksheet 3 (see **fig. 2**) captures the recorded data from worksheets 1 and 2 in a frequency chart and generates a column graph, so that students can easily compare outcomes from both flip sequences. By analyzing the generated representations, students are able to reason why getting exactly 5 heads and 5 tails in 10 flips of a fair coin is more likely than getting exactly 50 heads and 50 tails in 100 flips of a fair coin.

WORKSHEETS 1 AND 2

Before entering values and formulas, format worksheets 1 and 2 by inserting the appropriate text into cells **A3**, **C4**, **C5**, **C6**, and **D4** (see **fig. 3**). Note that the heading **# of Heads** is in a text box. To create a text box, click on the **Insert** tab, click on the **Text Box** button, and drag and drop the text box so that it is centered over cell **L1**; then enter the text. To change the color of the text box, select the text box, go to the **Home** tab, and select the drop-down arrow on the **Fill Color** icon to choose a color.

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To format the frequency table, highlight cells **C4–D6**, click on the **Home** tab, select the drop-down arrow on the **Border** icon, and click on **All Borders**. You may choose to alter the background color of the cells by selecting the cells, clicking on the **Home** tab, and clicking on the drop-down arrow on the **Fill Color** icon to choose a color.

Worksheets 1 and 2 are similar, except that worksheet 1 has 10 values in the **Results** column (representing a sequence of 10 flips) and worksheet 2 has 100 values in the **Results** column (representing a sequence of 100 flips). A description of how to create worksheet 1 follows; any code that differs for worksheet 2 is placed in braces. Following discussion of the initial setup of worksheets 1 and 2, details are provided for how to flip a sequence of coins, how to create a dynamically linked frequency chart and column graph of the outcomes, and how to code two macro buttons that record trial results.

To simulate flipping a fair coin, use the **RandBetween** function. **RandBetween** generates a random integer between the values specified in a given selection of cells. In cells **A4–A13** {**A4–A103**}, enter **=RandBetween(0,1)**. Use the **Fill Down** command (**Ctrl-D**) to populate these cells. Let 0 represent obtaining a head and let 1 represent obtaining a tail on a given flip.

To create a frequency chart of obtained results, use the **CountIf** function, which counts a designated entry within a selection of cells. In **D5**, type **=Countif(A4:A13,"0")** {**=Countif(A4:A103, "0")**} to count the number of heads obtained on the sequence of coin flips. In **D6**, type **=Countif(A4:A13,"1")** {**=Countif(A4:A103,"1")**} to count the number of tails obtained in the sequence of coin flips.

To create a dynamic visual of the total number of heads and tails obtained when flipping a coin 10 times, follow these steps: Highlight cells **C5–D6**, click on the **Insert** tab, click on the **Column Chart** button, and select **2-D Clustered Column**. To obtain the appropriate column **Label** menu, right-click on the inserted chart and choose **Select Data**. Under the **Horizontal (Category) Axis** labels, choose **Edit**. Highlight cells **C5** and **C6**, and then click on **OK**. You may add chart titles, axis titles, and data labels to the column chart by clicking on the chart and using the features located in the **Label** menu under the **Layout** tab.

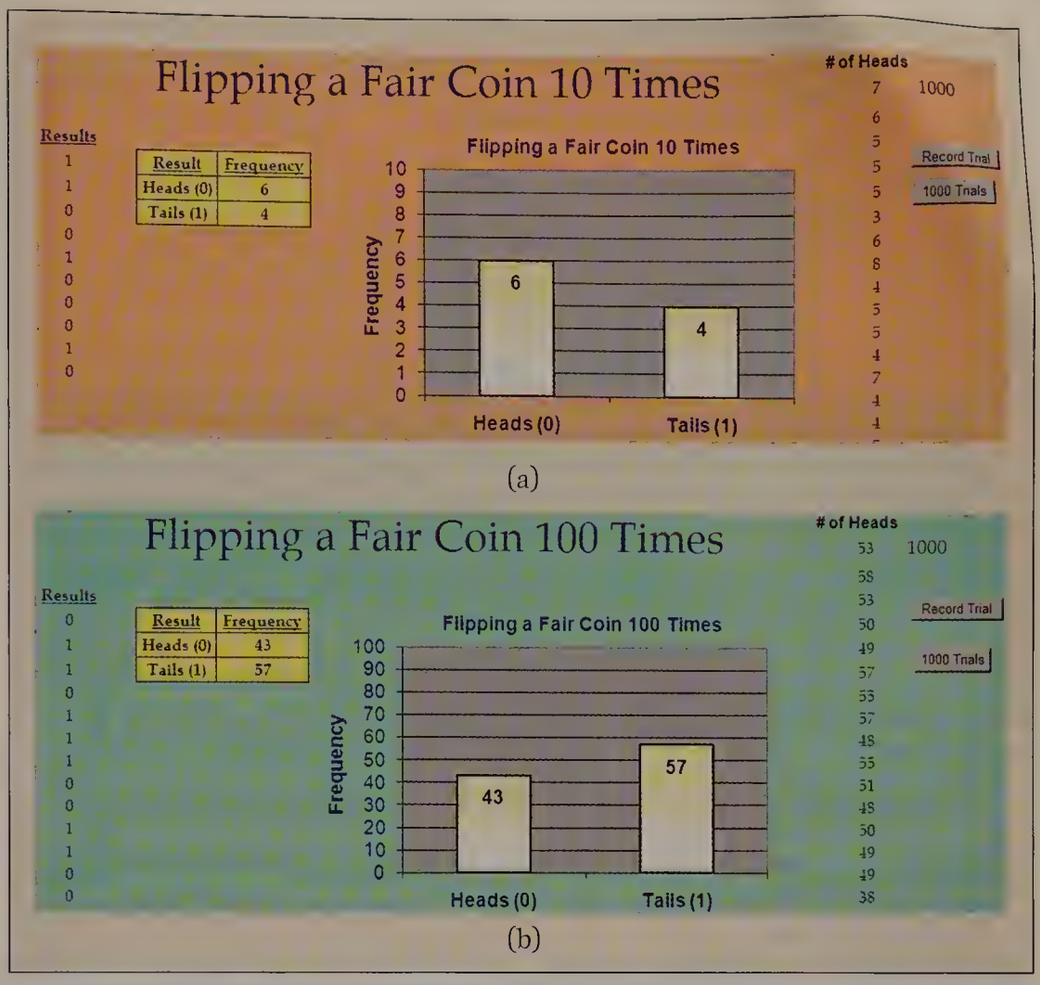

Fig. 1 Spreadsheets can be constructed to illustrate the coin-flipping experiment. The screen shots show the 1000th result of flipping a coin 10 times (**a**) and 100 times (**b**).

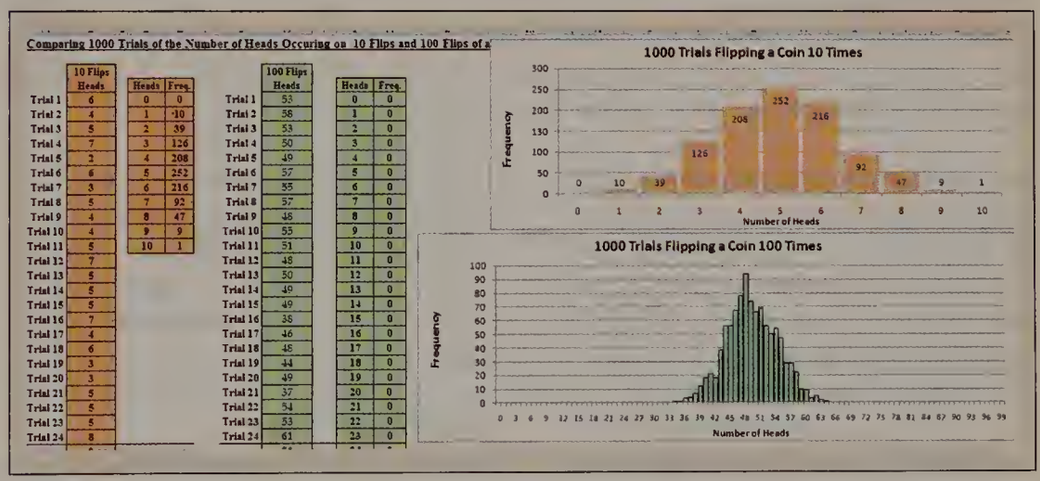
Comparing 1000 Trials of the Number of Heads Occuring on 10 Flips and 100 Flips of a
10 Flips Heads
100 Flips Heads
1000 Trials Flipping a Coin 10 Times
Frequency
Number of Heads
1000 Trials Flipping a Coin 100 Times
Frequency
Number of Heads

Fig. 2 Excel allows students to see multiple, dynamically linked representations of the data.

You may also choose to set the vertical axis so that the axis values remain fixed. To do so, click on the chart, click on the **Layout** tab, click on the **Axes** button, select **Primary Vertical Axis**, and choose **More Primary Vertical Axis Options**. In the **Axis Options**, set the **Maximum** to **10**, the **Minimum** to **0**, and the **Major Unit** to **1**.

When you press the recalculation key, typically **F9** or **CMD =**, Excel will simulate flipping a coin 10 times as well as capture and graph the number of heads and tails that occur in the sequence of these 10 flips. However, each time the recalculation key is pressed, the previous data set is erased and replaced by a new set.

Repeat these steps for worksheet 2.

For future coding, name the worksheets and workbook accordingly. Title

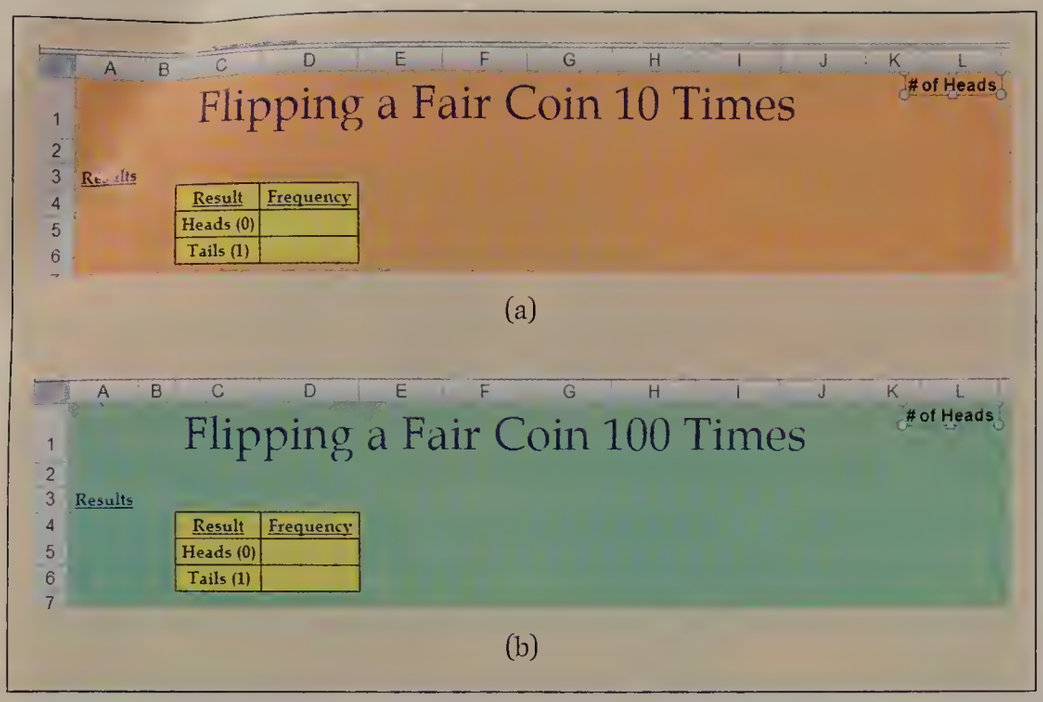

Fig. 3 Setting up worksheet 1 (**a**) and worksheet 2 (**b**) requires inserting a text box and formatting cells.

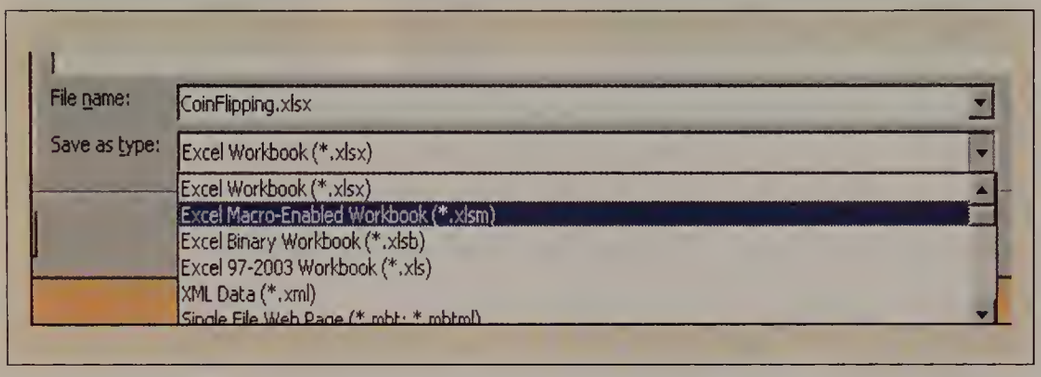

Fig. 4 Changing the workbook file type will allow the macros to save and run.

the workbook **CoinFlipping**, choose **Excel Macro-Enabled Workbook** (see **fig. 4**) in the drop-down box beside **Save as type** and click on **Save**. Name each worksheet according to its placement in the workbook. For example, the first worksheet will be called **Worksheet 1**.

Anytime an Excel macro-enabled workbook is opened, Excel will present a pop-up security warning stating that **Macros have been disabled** if the macro security setting is set to **Disable all macros with notification**. This setting can be found on the **Developer** tab in **Macro Security** under the **Macro Setting** menu. To enable the macros, select the **Options** button immediately to the right of the security warning, select **Enable this content**, and click on **OK**.

MACRO BUTTONS

To capture outcomes of multiple trials, you will create macro buttons using Visual Basic, a programming language built into Excel. A macro is a sequence of commands that is stored in a Visual Basic code file (a module). You will create macros and buttons linked to them so that, on clicking, the respective macros will be activated. These buttons are created by writing code directly into the module. The two buttons you create are **Record Trial** and **1000 Trials**. The **Record Trial** button records the total number of heads that occurred in one trial. The **1000 Trials** button records the total number of heads that occurred in each of 1000 trials.

To create a button that runs a macro, place the **Developer** tab in the Ribbon. Click on the **Microsoft Office** button and select **Excel Options**. (Some users may have to select **Prepare** before choosing **Excel Options**.) Choose **Popular Menu** and check **Show Developers Tab in the Ribbon** (see **fig. 5**). Now select **Formulas** from the **Options Menu** at the left. Select **Manual** under **Calculation Options**, uncheck the box that says **Recalculate workbook before saving** (see **fig. 6**), and then click on **OK**.

RECORD TRIAL BUTTON

To insert a button, click **Insert** under the **Developer** tab and select the **Button** icon in **Form Controls** (see **fig. 7**). On clicking on the **Button** icon, drop the button into your worksheet. The **Assign Macro** window will pop up. Clicking on **New** will open a module in Visual Basic. In this window, type the code in **figure 8**. This code accesses the total number of heads value that is located in cell **D5** and places it into the first available cell (moving downward in **Column L**). To ensure that a value in **Column L** is never replaced, cell **M1** is used as a counter that is updated each time the button is pressed. This counter also keeps track of the number of times you have recorded a value in

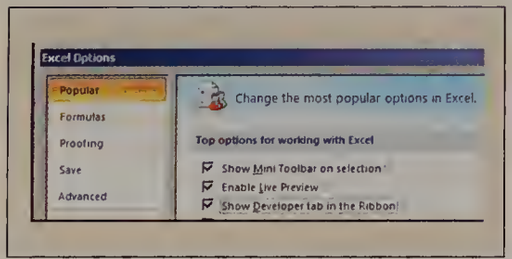

Fig. 5 Excel enables students to display the developer's tab in the ribbon.

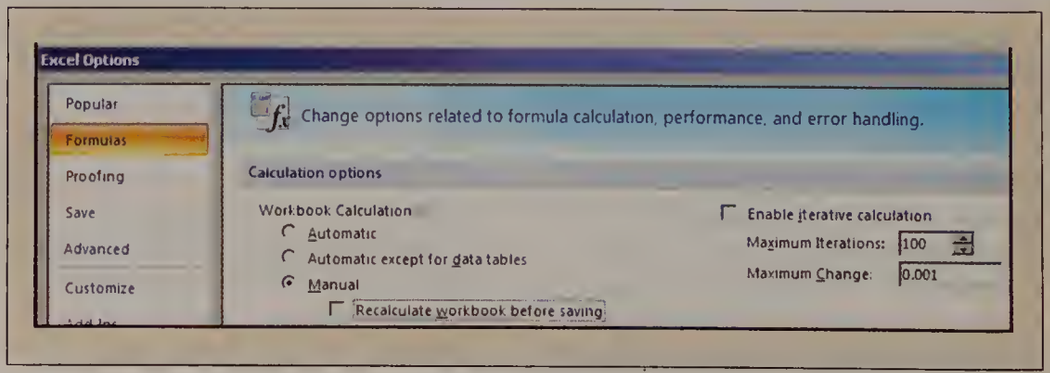

Fig. 6 Excel allows students to choose automatic or manual calculation.

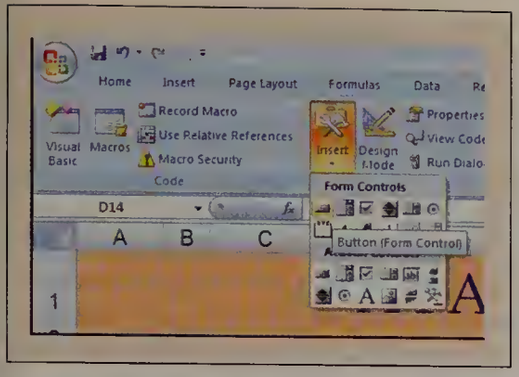

Fig. 7 To activate the macro code, insert a button from the **Form Controls** menu.

```
(General)
Sub Button1_Click()

Dim locArray(1 To 2) As String

Calculate
locArray(1) = "L"
locArray(2) = Range("M1").Value
Location = Join(locArray, "")
Range(Location).Value = Range("D5").Value

End Sub
```

Fig. 8 This code programs Excel to capture and record the number of heads obtained from a trial.

```
Sub Button1_Click()

Dim locArray(1 To 2) As String

Calculate
locArray(1) = "L"
locArray(2) = Range("M1").Value
Location = Join(locArray, "")
Range(Location).Value = Range("D5").Value

End Sub
Sub Button2_Click()

Dim d As Integer
    Columns("L:L").Select
    Selection.ClearContents

For d = 0 To 999
    Application.Run "CoinFlipping.xlsm!Button1_Click"

Next d

End Sub
```

Fig. 9 This code programs Excel to capture and record the number of heads obtained from each of 1000 trials.

```
Sub Button1_Click()

Dim locArray(1 To 2) As String

Calculate
locArray(1) = "L"
locArray(2) = Range("M1").Value
Location = Join(locArray, "")
Range(Location).Value = Range("D5").Value

End Sub
Sub Button2_Click()

Dim d As Integer
    Columns("L:L").Select
    Selection.ClearContents

For d = 0 To 999
    Application.Run "CoinFlipping.xlsm!Button1_Click"

Next d
End Sub

Sub Worksheet2_Button1_Click()

Dim locArray(1 To 2) As String

Calculate
locArray(1) = "L"
locArray(2) = Range("M1").Value
Location = Join(locArray, "")
Range(Location).Value = Range("D5").Value

End Sub

Sub Worksheet2_Button2_Click()

Dim d As Integer
    Columns("L:L").Select
    Selection.ClearContents

For d = 0 To 999
    Application.Run "CoinFlipping.xlsm!Worksheet2_Button1_Click"

Next d
End Sub
```

Fig. 10 The code for all four buttons is entered into one Visual Basic module.

Column L. Save your code (**Ctrl S**) and return to the spreadsheet.

To program the counter, type **=1+CountA(L1:L1000)** in cell **M1**. By using the range **L1–L1000**, **M1** will count a maximum of 1000 simulations.

1000 TRIALS BUTTON

The purpose of the **1000 Trials** button is to eliminate the need to click on the **Record Trial** button 1000 times to obtain results for 1000 trials. To create the **1000 Trials** button, insert a **Form Controls** button from the **Developer** tab and drop it into your worksheet. After clicking on **New**, you will see the code of your first button. Below **Sub Button2_Click()**, enter the code shown in **figure 9**. This code first removes all previous recorded values in column **L**. Surrounding the **Application.Run** statement, which activates the **Record Trial** code, with a **For-To** loop simulates flipping a coin 10 times and records the number of heads obtained in each of the 1000 trials. Save your code (**Ctrl S**), return to Excel, and name this button **1000 Trials**.

You have now completed worksheet 1 and can repeat the directions to create worksheet 2. Note that each time you introduce a button, it will be given a name **Button#_Click()** in which the number (**#**) will increase. When you code the **1000 Trials** button in worksheet 2, make sure that your **Application.Run** statement refers to the **Record Trials** button in worksheet 2. **Figure 10** shows all four codes in Visual Basic Editor.

WORKSHEET 3

Worksheet 3 links the information from the two previous worksheets to create frequency charts and column graphs of the total number of heads obtained from 1000 trials of flipping a coin 10 times and 100 times (both shown in **fig. 1**). Before linking values, format

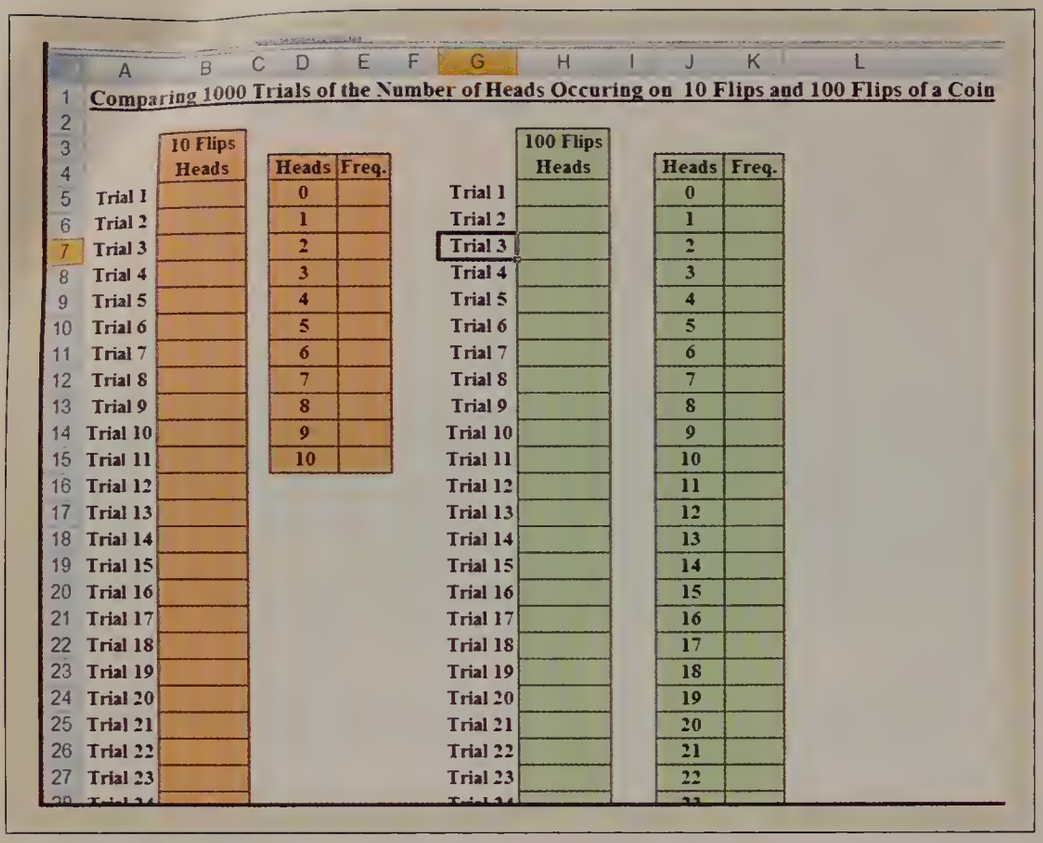

Fig. 11 Setting up worksheet 3 requires formatting cells.

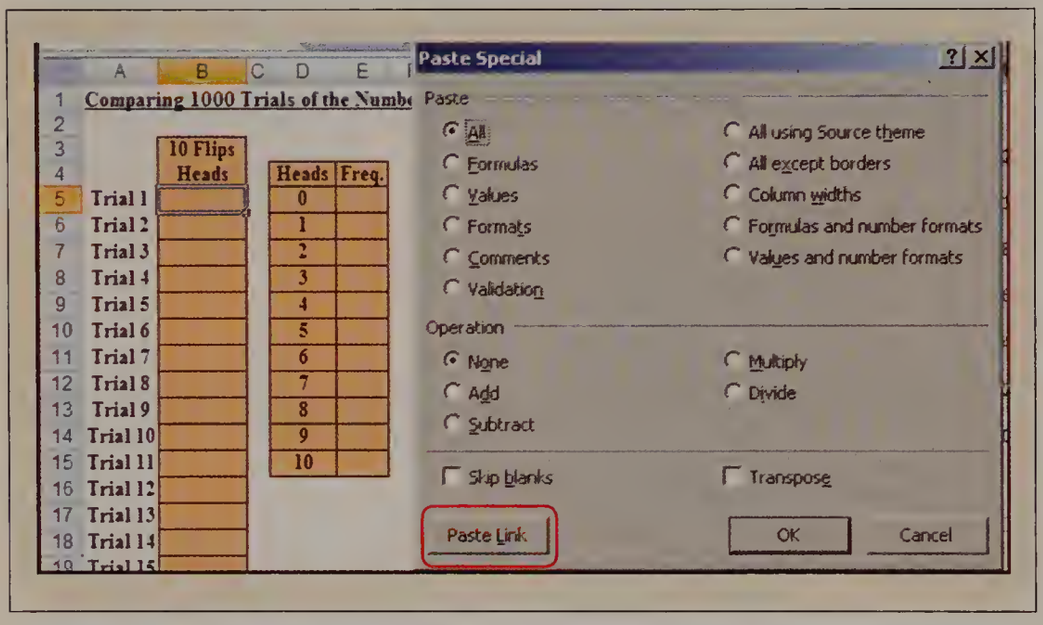

Fig. 12 Excel allows students to link data among worksheets.

worksheet 3 by inserting text into the appropriate cells (see **fig. 11**) and typing the numbers 0–10 in cells **D5–D15** and the numbers 0–100 in cells **J5–J105**. To put borders around selected cells, follow these directions: Highlight cells **B3–B104**, **D4–E15**, **H3–H104**, and **J4–K105**; go to the **Home** tab; select the drop-down arrow on the **Border** icon; and select **All Borders**. You may choose to color the cells according to your worksheet 1 and worksheet 2 background colors (see **fig. 11**).

Return to worksheet 1, highlight cells **L1–L1000**, and copy them. In worksheet 3, select cell **B5**, right-click, select **Paste Special**, and click on the **Paste Link** button (see **fig. 12**). Now the heads obtained from each trial are linked to worksheet 3; therefore, whenever the **1000 Trials** button in worksheet 1 is pressed, the data will update not only in worksheet 1 but also in worksheet 3. To link the data from worksheet 2 to worksheet 3, highlight the cells **L1–L1000** in worksheet 2 and copy them. In worksheet 3, highlight cell **H5**, right-click, select **Paste Special**, and click on the **Paste Link** button.

Now, students will create frequency charts and column graphs, as shown in **figure 2**. To create a frequency chart for worksheet 1 data, highlight **E5–E15** in worksheet 3; type **=Frequency(B5:B1004, D5:D15)** in the formula bar; and press and hold the following keys in order: **Ctrl, Shift, Enter**. To create a frequency chart for worksheet 2 data, highlight **K5–K105** in worksheet 3; type **=Frequency(H5:H1004, J5:J105)** in the formula bar; and then press and hold the following keys in order: **Ctrl, Shift, Enter**. With these frequency charts, you can now insert a column graph for each data set.

To create a column graph for the 1000 trials of flipping a coin 10 times, highlight cells **E5–E15**, click on the **Insert** tab, click on the **Column Chart** button, and select **2-D Clustered Column**. Now, in the graph, right-click on a column in the column graph. Click on **Select Data**. In the new screen, under **Horizontal (Category) Axis Labels**, click on **Edit**. Highlight **D5–D15**. Click **OK** twice.

To create a column graph for the 1000 trials of flipping a coin 100 times, highlight cells **K5–K105**, click on the **Insert** tab, click on the **Column Chart** button, and select **2-D Clustered Column**. Now, in the graph, right-click on a column in the column graph. Click on **Select Data**. In the new screen, under **Horizontal (Category) Axis Labels**, click on **Edit**. Highlight **J5–J105**. Click **OK** twice. You may add a title, labels, and fixed vertical axis values by using the features in the **Layout** tab (see **fig. 2**).

PUTTING THE WORKSHEETS INTO PRACTICE

Before using this workbook in the classroom, delete the entries under the **Results** column and in cells **L1–L1000** in both worksheets 1 and 2. This procedure allows students to enter the **RandBetween** function under the **Results** column, collect data, and view the dynamic representations.

As students run the simulation, have them answer the following questions:

1. Out of 1000 trials, what was the largest discrepancy between the

number of heads and the number of tails obtained within one trial for a sequence of 10 flips? For a sequence of 100 flips?

2. Out of 1000 trials, were there any results of heads obtained that appeared to occur more often than others within the sequence of 10 flips? Within the sequence of 100 flips?
3. For the 10 flips sequence, what is the total number of heads that occurred over 1000 trials? What is the average number of heads that occurred?
4. For the 100 flips sequence, what is the total number of heads that occurred over 1000 trials? What is the average number of heads that occurred?
5. Which of the following has a higher probability—getting exactly 5 heads and 5 tails in 10 flips of a fair coin or getting exactly 50 heads and 50 tails in 100 flips of a fair coin?
6. Which of the following has a higher probability—getting 40 to 60 percent heads in 10 flips of a fair coin or getting 40 to 60 percent heads in 100 flips of a fair coin?

At the end of the activity, have students discuss their initial predictions, the process and results of the simulations, their answers to and reasoning about the questions above, and their understanding of the law of large numbers.

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Editor's note: See also Laurie H. Rubel, "Connecting Research to Teaching: Is 7/10 Always Equivalent to 700/1000?" *MT* September 2010 (vol. 104, no. 2, pp. 144–47).

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Dick J. Smith and Eric F. Errthum

Finding Skewed Lattice Rectangles

Many mathematics instructors attempt to insert guided exploration into their courses. However, exploration tasks frequently come across to students as contrived, pertinent only to the most recently covered section of the textbook. In addition, students usually assume that the teacher already knows the answers to these explorations.

The true benefit of guided exploration comes when students develop a real sense of ownership of a problem, work on it outside class, and see the teacher as a companion on their journey of exploration. Such was the case with the undergraduate modern geometry class taught by coauthor Smith. We will show how a specific problem intended to teach a specific concept grew into a larger exploration, with connections between algebra and geometry.

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

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THE PROBLEM

Ask carpenters how they know whether the corners of the rectangular frame they constructed are "square," and they will respond that they simply check that the diagonals have the same measure. This ancient concept, which predates the ideas of analytic geometry and slope, can be stated as a theorem of geometry: If the diagonals and both pairs of opposing sides of a quadrilateral are congruent, the quadrilateral is a rectangle.

This geometry theorem was the topic at hand the day Smith asked his students to verify that the four coordinates M (5, –7), N (1, –13), O (28, –31), and P (32, –25) were the vertices of a rectangle (see **fig. 1**). Students were familiar with the distance formula and computed the lengths of the four sides and the two diagonals. Smith had used this problem for many years, but this time something different happened. After class that day, a student stopped by to discuss the problem. Tony's curiosity was piqued when he checked the diagonal measures and found that

$$MO = \sqrt{23^2 + 24^2}$$
$$= NP = \sqrt{31^2 + 12^2} = \sqrt{1105}.$$

He did not expect to see two different pairs of squared integers yield the same number. By reflecting on the solution of a routine problem, Tony became motivated to explore the problem further.

Pólya (2004) is explicit about the opportunities for learning mathematics evoked by this simple act of "thinking back." He writes: "Even fairly good students, when they have obtained the solution of the problem and written down neatly the argu-

ment, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work" (Pólya 2004, p. 14). In the journey we are about to describe, we came to understand more fully what Pólya meant when he said, "A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted" (Pólya 2004, p. 15).

THE ACTIVITY

Mathematics teachers become enthusiastic when students become enthusiastic. Tony tried to find other sets of four numbers that would solve $a^2 + b^2 = c^2 + d^2$. The first numbers he found were $a = 1$, $b = 7$, $c = 5$, and $d = 5$. He then attempted to find vertices at lattice points that produced a skewed rectangle with diagonals that measured $\sqrt{50}$. Although Tony had trouble with this task, he was tenacious. His persistence encouraged Smith to start playing around with the problem and share it with coauthor Errthum.

Together we decided to build a series of conversations with Smith's students to guide them down a path of discovery. We wanted students to think about the geometry related to the integer solutions of $a^2 + b^2 = c^2 + d^2$. We wanted them to think deeply, for a longer period of time than we could devote to this problem in class. So we developed a set of prompts to extend the investigation into rectangles and encourage students to discover strategies for finding integer solutions to the equation $a^2 + b^2 = c^2 + d^2$.

Analytic geometry allows students to perform calculations and algebraic manipulations related to geometric deductions and conjectures. However, the quantities are often fractional or irrational. Some students have difficulty finding rectangles whose vertices are at lattice points—that is, points with integer coordinates—unless the sides of the rectangles are parallel to axes. In our guided exploration, we stipulated that the vertices of the rectangles be lattice points.

Our question and the answers it evoked engaged students in the kind of mathematical thinking called for in the Standards for Mathematical Practice listed in the Common Core State Standards Initiative (CCSSI 2010). In particular, this exploration engaged students in using specific examples to formulate generalizations and make use of algebraic structure to model geometric situations. As students brought their own discoveries to class, they learned to explain their work and criticize others' work productively.

By design, the investigation supplemented topics covered in our regular geometry classes. Our hints and student conjectures would span months. We asked students to immerse themselves in a problem that required a long time and a diligent attitude. In this way, students would have the chance to take ownership of the problem and realize the nature of true mathematical investigation.

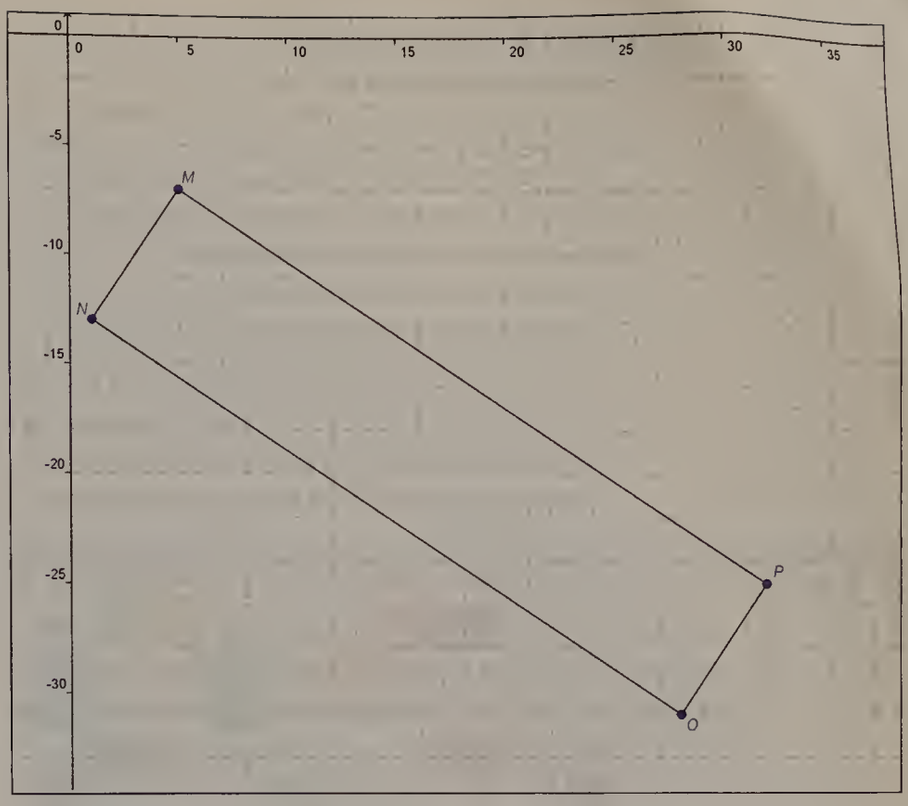

Fig. 1 Students were given the following task: Verify that this quadrilateral is a rectangle. The answer produces an interesting equality.

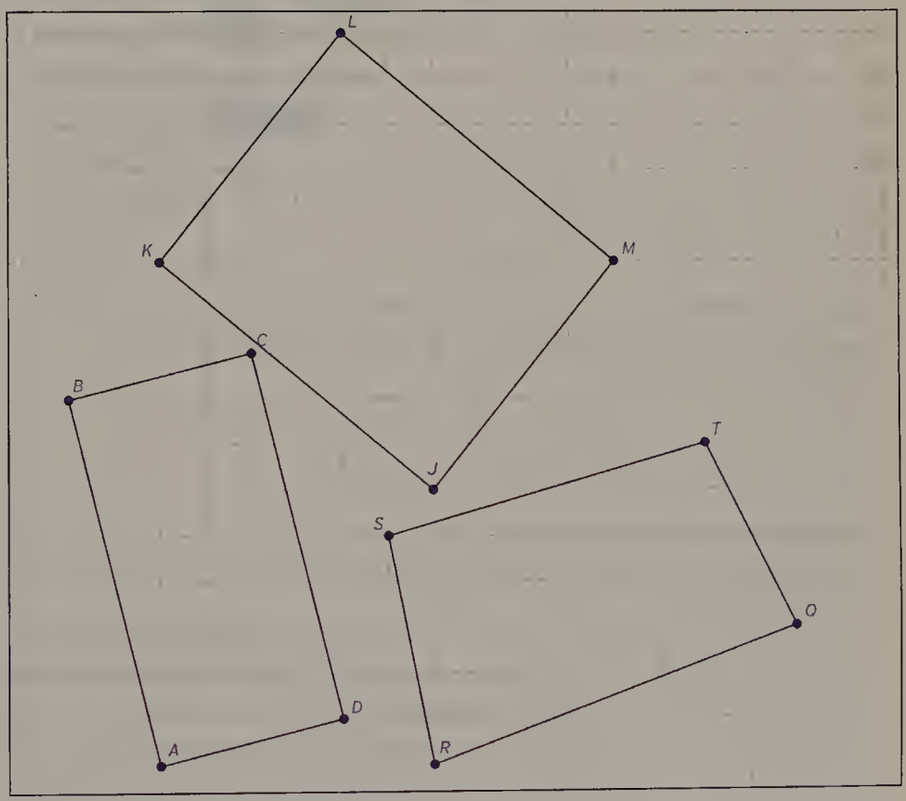

Fig. 2 Which of these quadrilaterals are rectangles?

To initiate the journey in the classroom, Smith presented three quadrilaterals (see **fig. 2**) and asked students to determine which are rectangles. He reminded students that quadrilaterals with equal diagonals and equal opposite sides are rectangles.

Naturally, students decided to focus mainly on the diagonal measures and see what they could discover. In their initial discussion of the problem, students decided against considering slope, trigonometry, or other tools of analytical geometry, but they agreed to use the distance formula and the basic algebra of the integer coordinate plane.

During the next class, we discussed students' work. Most had done a good job of using the distance formula to show that quadrilateral $QRST$ is not a rectangle. Even though the diagonals are both equal to $\sqrt{85}$, the sides are not equal. Randi said, "I just looked and knew that it was not a rectangle."

That quadrilateral $JKLM$ is not a rectangle is not as visually obvious. Although the two pairs of sides are equal, respectively, to $\sqrt{41}$ and $\sqrt{61}$, the diagonals are $LJ = \sqrt{104}$ and $MK = 10$. Randi mentioned that she had to do the mathematics to examine this one.

Students were pleased to find that quadrilateral $ABCD$ is a rectangle, even though its sides are not parallel to the x- and y-axes. This quadrilateral passes the test because the two pairs of sides are equal, respectively, to $\sqrt{68}$ and $\sqrt{17}$ and the diagonals are both equal to $\sqrt{85}$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	5	10	17	26	37	50	65	82	101	122	145	170	197
2		8	13	20	29	40	53	68	85	104	125	148	173	200
3			18	25	34	45	58	73	90	109	130	153	178	205
4				32	41	52	65	80	97	116	137	160	185	212
5					50	61	74	89	106	125	146	169	194	221
6						72	85	100	117	136	157	180	205	232
7							98	113	130	149	170	193	218	245
8								128	145	164	185	208	233	260
9									162	181	202	225	250	277
10										200	221	244	269	296
11											242	265	290	317
12												288	313	340
13													338	365
14														392

Fig. 3 Students use Excel to find instances of $a^2 + b^2 = c^2 + d^2$.

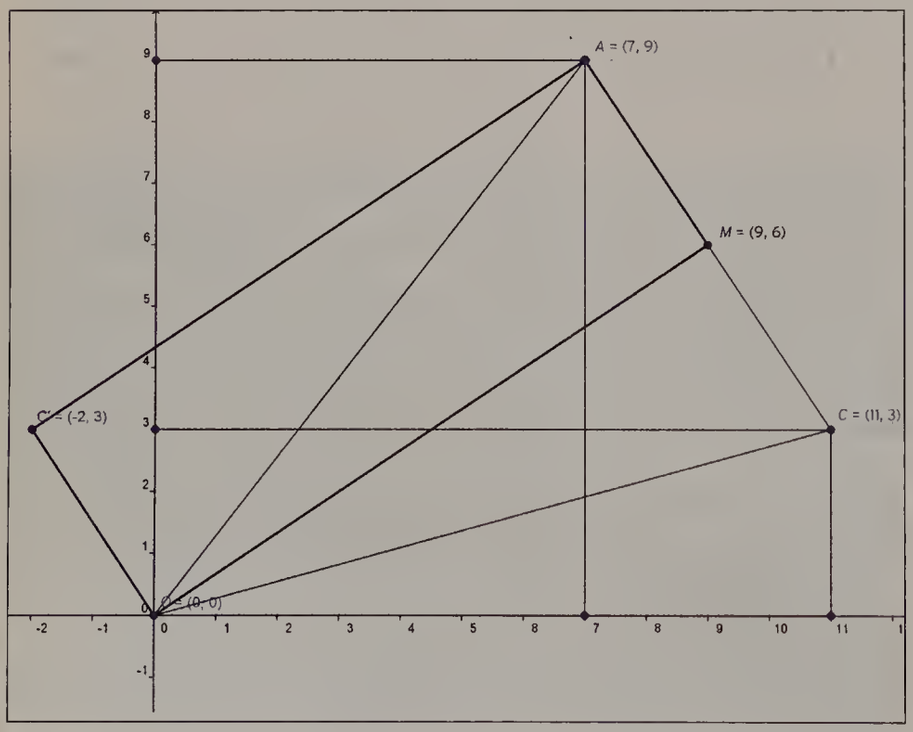

Fig. 4 Stacking congruent right triangles produces a skewed rectangle.

OTHER STUDENTS JOIN TONY'S PURSUIT

A week had now passed, and Tony had still not found lattice vertices of a skewed rectangle with a diagonal of $\sqrt{50}$. Also, he was frustrated that he could not find many integer solutions to

$$\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}.$$

We encouraged other students to help in the search.

Our students are required to be proficient with spreadsheets. Without prompting, Tracy developed a spreadsheet (see **fig. 3**) to investigate the values of $a^2 + b^2$. Using the algebraic commands of the spreadsheet, she set the top row and the right column to be whole numbers and the cells to be the sum of the square of these whole numbers. She then visually inspected the spreadsheet to find solutions to the equation $a^2 + b^2 = c^2 + d^2$.

With the discovery of a method for generating numerous integer solutions to $a^2 + b^2 = c^2 + d^2$, Tony and Tracy joined forces and attempted to form skewed rectangles whose diagonal measures corresponded to those integer solutions. They decided to start with the cells for $7^2 + 9^2$ and $11^2 + 3^2$ (those containing 130 in **fig. 3**). They drew the two rectangles having sides on the x- and y-axes and corners at the points A (7, 9) and C (11, 3), respectively.

Tracy drew $\overline{AC}$ and found the midpoint M (9, 6) (see **fig. 4**). Tracy and Tony noticed that triangles MOA and MOC were congruent by SSS. They could now stack these two congruent right triangles to form a skewed rectangle with vertices at lattice points and diagonals of length $\sqrt{130}$. The fourth point they constructed was C' (−2, 3). When they showed their results to Smith, he challenged them to generalize this approach by using the variables a, b, c, and d. Three days later, Tracy demonstrated the following for A (a, b) and C (c, d) where $a^2 + b^2 = c^2 + d^2$:

$$OA = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

$$C'M = \sqrt{\left(\frac{a+c}{2} - \frac{a-c}{2}\right)^2 + \left(\frac{b+d}{2} - \frac{b-d}{2}\right)^2}$$

$$= \sqrt{c^2 + d^2}$$

Tony also noticed that he could create many

rectangles at different lattice points in the plane by translating the one that he and Tracy had discovered and also by stacking the smaller rectangles to create a larger one (see **fig. 5**). They enjoyed demonstrating these rectangles to the class, and everyone seemed to want to investigate the problem in more depth.

Several days later, Kelli came to class with another discovery. She had used Tracy's coordinates,

$$\left(\frac{a+c}{2}, \frac{b+d}{2}\right) \text{ and } \left(\frac{a-c}{2}, \frac{b-d}{2}\right),$$

and rearranged their y-coordinates to get

$$\left(\frac{a+c}{2}, \frac{b-d}{2}\right) \text{ and } \left(\frac{a-c}{2}, \frac{b+d}{2}\right),$$

noting that this rearrangement does not change the computation for the length $C'M$. The rearrangement created points (–2, 6) and (9, 3). Kelli still used (0, 0) and $(a, b) = (7, 9)$ as the two other vertices. Taking these ideas another step forward, she then wrote

$$\left(\frac{a+c}{2}, \frac{d-b}{2}\right) \text{ and } \left(\frac{c-a}{2}, \frac{b+d}{2}\right)$$

and plotted (9, –3) and (2, 6) along with (0, 0) and $(c, d) = (11, 3)$. Her rectangles were not developed by stacking previous rectangles on top of each other, but both had diagonals that measured

$$\sqrt{7^2+9^2} = \sqrt{11^2+3^2} = \sqrt{130}.$$

Kelli then drew a circle with center (0, 0) containing points (7, 9) and (11, 3) and challenged other students to look for rectangles with vertices at lattice points intersected by the circle (see **fig. 6**).

At this juncture, students were becoming excited. Karissa took one of Kelli's rectangles and started stacking copies of it to the right and up. **Figure 5** leads to the following integer solutions:

$$22^2 + 6^2 = EI = AG = 14^2 + 18^2$$
$$20^2 + 0^2 = EH = DG = 16^2 + 12^2$$
$$13^2 + 9^2 = EB = AF = 5^2 + 15^2$$

Students started using Karissa's technique to produce other examples of the algebraic phenomenon $a^2 + b^2 = c^2 + d^2$. But they noticed that they still needed an original skewed rectangle to start the process. This was an opportune moment to move students away from the spreadsheet-motivated solution to a more efficient and general method for finding solutions to $a^2 + b^2 = c^2 + d^2$.

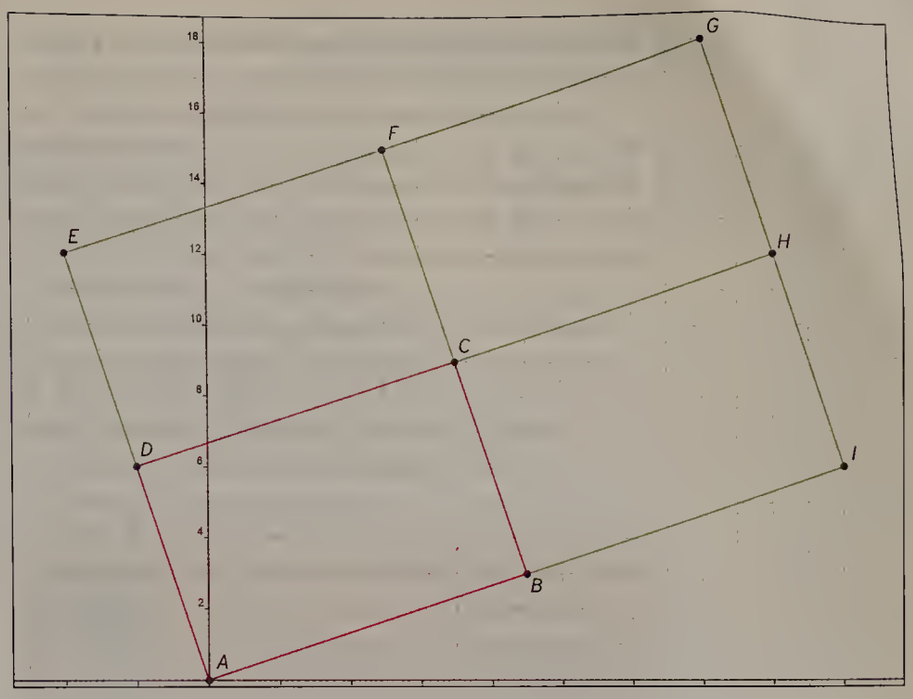

Fig. 5 Translations of the original (red) rectangle will produce others.

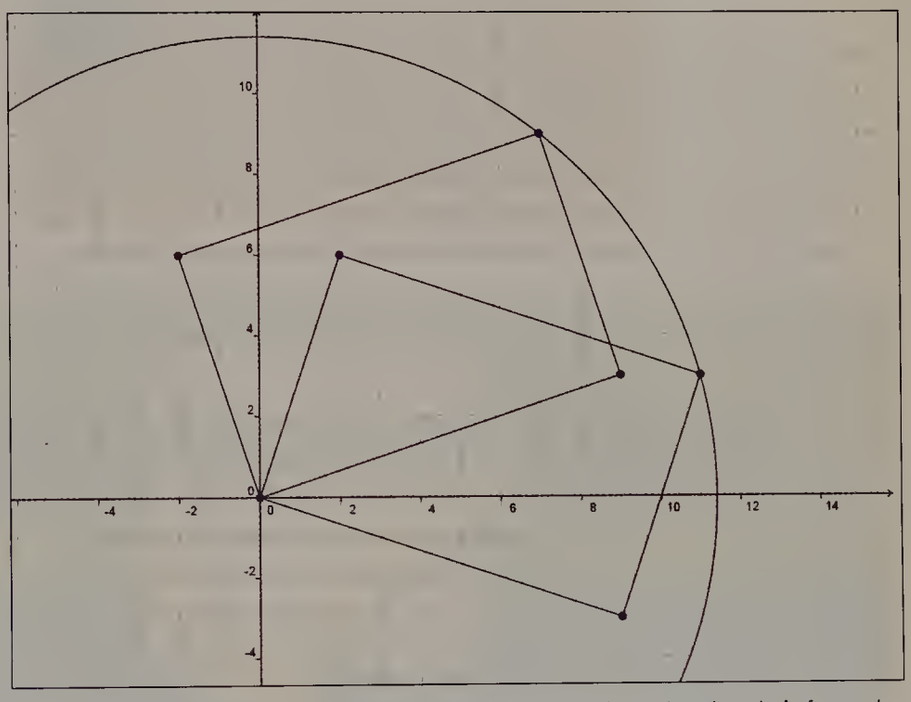

Fig. 6 When the circle intersects a lattice point, a new skewed rectangle is formed.

HELPFUL ALGEBRA

Students were now ready to be taught the algebra that fit the problem we were investigating. We introduced them to the following method of creating a, b, c, and d that satisfy $a^2 + b^2 = c^2 + d^2$. This technique comes from rearranging the equation into $a^2 - c^2 = d^2 - b^2$ and then factoring to obtain $(a + c)(a - c) = (d + b)(d - b)$. Hence, a clue for finding a, b, c, and d is to study numbers that have two different factorizations.

Choose a positive whole number N that can be written as a product of two distinct whole numbers in two different ways: $N = ef = gh$. (For simplicity, to generate positive solutions, choose $e > f$ and $g > h$.) Now, for each of these pairs of factors,

set $a = e + f$ and $c = g + h$ and then set $d = e - f$ and $b = g - h$, making $a^2 + b^2 = c^2 + d^2$.

In general, for any e, f, g, and h that satisfy $ef = gh$, the following algebraic derivation provides a direct proof of the method:

$$ef = gh$$
$$4ef = 4gh$$
$$2ef - 2gh = 2gh - 2ef$$
$$e^2 + 2ef + f^2 + g^2 - 2gh + h^2 = g^2 + 2gh + h^2 + e^2 - 2ef + f^2$$
$$(e + f)^2 + (g - h)^2 = (g + h)^2 + (e - f)^2$$
$$a^2 + b^2 = c^2 + d^2$$

As an example, choose $N = 8 = 8 \cdot 1 = 4 \cdot 2$. This leads to $a = 8 + 1 = 9$; $b = 4 - 2 = 2$; $c = 4 + 2 = 6$; and $d = 8 - 1 = 7$.

CONTINUING THE EXPLORATION

Apparently, the revelation of an algebraic method for generating integer solutions to the equation $a^2 + b^2 = c^2 + d^2$ was not enough. Students began meeting outside class to discuss the problem. Six class members worked with the method presented above using $N = 144$. They found 28 different solutions for $a^2 + b^2 = c^2 + d^2$. They also discovered that $a^2 + b^2 = c^2 + d^2 = 25{,}925$ was largest when $N = 1 \cdot 144 = 2 \cdot 72$ and that $a^2 + b^2 = c^2 + d^2 = 625$ was the smallest when $N = 9 \cdot 6 = 12 \cdot 12$. Randi noticed that one-fourth of the sums were themselves squares: 625, 676, 900, 1600, 2601, 5476, and 21025. She further noticed that they had all occurred when one of the factored forms was $12 \cdot 12$. Without realizing it, Randi had stumbled on a method for generating Pythagorean triples: Let $N = g^2$, a perfect square with factorization $ef = gg$, and apply the method given above.

At this point, students were pleased with the state of the problem. Eventually, however, the most industrious students started looking for shortcuts. They wanted to know if there were a way simply to substitute numbers. So we showed them a more general method.

Consider N in terms of its prime factorization. Expressing N as the product of two numbers is essentially taking some prime factors to make a "left" factor and using the rest to make a "right" factor. To factorize N as a different product, some prime factors just have to swap sides. Algebraically, this formulation looks like

$$N = (wx)(yz) = (wy)(xz)$$

where each quantity in parenthesis represents a "left" or "right" factor and x and y are the parts they swap to make a new factoring of N. (Of course, this means that x cannot equal y and w cannot equal z but some of the variables may equal 1.)

Using this thought process together with our algorithm above produces the following:

$$a = wx + yz$$
$$b = wy - xz$$
$$c = wx - yz$$
$$d = wy + xz$$

A little algebra then reveals that

$$a^2 + b^2 = (wx)^2 + (wy)^2 + (xz)^2 + (yz)^2 = c^2 + d^2.$$

We then asked students to choose four numbers at random. They chose $w = 3$, $x = 5$, $y = 4$, and $z = 8$ and then computed $a = 47$, $b = -28$, $c = -17$, and $d = 52$ to produce the identity $47^2 + (-28)^2 = (-17)^2 + 52^2$. We returned to the geometrical aspect of the problem and used these values to create the rectangle with vertices at (0, 0), (15, 12) = $((a + c)/2, (b + d)/2)$, (–17, 52), and (–32, 40) = $((c - a)/2, (d - b)/2)$.

We all had now arrived at the point at which we could produce as many examples of $a^2 + b^2 = c^2 + d^2$ as we wanted just by substituting new numbers. Similarly, we could create as many skewed lattice rectangles as we wanted.

ADDING ANOTHER TERM

Of course, once one hits the jackpot, greed sets in. Now the authors quickly began to ask themselves how to find solutions to $a^2 + b^2 + c^2 = d^2 + e^2 + f^2$. Neither of us could readily arrive at a simple technique, so we offered this problem as a challenge to the students. We confessed that we did not know the answer either, and for a while everyone—students and instructors—were stumped.

Smith later realized that the same factorization technique used earlier could be applied in this situation. If $a^2 + b^2 + c^2 = d^2 + e^2 + f^2$, then $a^2 - d^2 + b^2 - e^2 = f^2 - c^2$. Hence,

$$(a + d)(a - d) + (b + e)(b - e) = (f + c)(f - c).$$

Smith stopped here and asked students what they would do now. They randomly chose integers a, d, b, and e to compute the sum on the left side. They wrote this sum as a product of two factors and solved for the necessary f and c to complete the identity. (Note: In general, depending on our choices for a, d, b, and e, we may have to multiply everything by 2 to ensure that f and c turn out to be integers.)

As an example, let $a = 7$, $d = 2$, $b = 6$, and $e = 3$. Then $(9)(5) + (9)(3) = 72 = (f + c)(f - c)$. If we factor 72 as $18 \cdot 4$, solve $f + c = 18$ and $f - c = 4$ to arrive at $f = 11$ and $c = 7$. Therefore, $7^2 + 6^2 + 7^2 = 3^2 + 2^2 + 11^2$. Alternatively, we could have factored 72 as $36 \cdot 2$ and arrived instead at $7^2 + 6^2 + 17^2 = 3^2 + 2^2 + 19^2$.

The students got excited about the additional challenge of adding a term. Without teacher input, they made three-dimensional boxes and measured the oblique diagonals. They considered this extension as belonging to us—teachers and students. They claimed ownership because they were involved in the whole project.

A POSTSCRIPT

This experience was more profoundly beneficial for students than any of the other regular exercises or activities. Because students considered this to be *their* problem, they listened to our input and were motivated to learn. Tracy, now a middle school teacher, says that her students can follow what she is doing when using a simplified version of the same problem. During her student teaching, Karissa showed her students what she called "her problem." Kelli and Randi plan to incorporate this problem in their preservice teaching.

Our students learned that deep mathematical understanding develops over time. They now see mathematics as something that can be long and messy. However, with time, insight, and teamwork, interesting connections can sometimes be found.

As teachers, we found that not knowing the answers all the time actually helped students. We listened to their questions, communicated with them, and, when the moment was right, shared some of our thoughts with them.

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All votes must be received by October 31st. Be heard and cast your vote today!

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CALL FOR MANUSCRIPTS

2014 Focus Issue

What Makes Ideas Stick?

Teachers design lessons to engage students with big ideas of mathematics. Does engagement necessarily imply that students will retain what we hope they will retain? How do students in your classes develop an appreciation for mathematical big ideas and the relationships among them? What lesson characteristics seem to promote the permanence of concepts in the minds of your students?

What works? Write about your big ideas—*big* big ideas or *small* big ideas—that help make mathematics stick. Following are a few suggestive, but not prescriptive, prompts:

- Great lessons stay in the minds of students. What lessons do you have that engage your students in exciting and meaningful mathematical experiences?
- Some ideas reside with students because of the instruction that introduces them more than the inherent mathematics. Do you start with an intriguing idea and then develop it throughout class, or do the big ideas build in the background?
- Students often surprise us with their own "aha!" moments, usually as they experience the epiphany of an unexpected connection. What are these moments, and how do we foster these connections and epiphanies?
- How do reasoning and sense making evolve to foster mathematical memory? What are high-leverage practices that help ideas stick?
- What kinds of challenges do big mathematical ideas face in the current political climate?
- How can technology impact the "stickiness" of ideas?
- How do we ensure that students internalize the mathematics that they need to be successful in subsequent courses? How do they develop the mathematical sophistication to adapt to an ever-changing world?

Contribute to this special *MT* issue by highlighting mathematical and pedagogical big ideas from your own teaching. Show us what helps students retain ideas and what leads them to an appreciation of broad concepts.

The *MT* Editorial Panel invites submissions for the 2014 Focus Issue. Please submit manuscripts at **mt.msubmit.net** by **May 1, 2013**. Be sure to check the box indicating that this manuscript is for the 2014 Focus Issue entitled What Makes Ideas Stick? Guidelines for the preparation of manuscripts can be found at www.nctm.org/publications/content.aspx?id=22602. No author identification should appear in the manuscript's article file. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Albert Goetz, agoetz@nctm.org.

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MATHEMATICS teacher

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FOR YOUR information

PUBLICATIONS

Create Success! Unlocking the Potential of Urban Students, Kadhir Rajagopal, 2011. 147 pp., $24.95 paper. ISBN 978-1-4166-1113-4. ASCD; www.ascd.org.

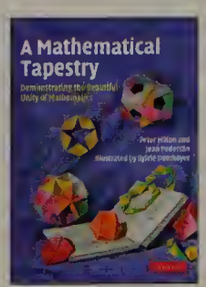
This book, as the author states, is intended to help teachers help their students achieve a higher level of success. Designed primarily for teachers working with middle school and high school students, the CREATE model is based on three essential principles that influence student learning: quality instruction, the classroom environment (i.e., the relationship between teachers and their students), and a mutual sense of accountability.

Having spent twenty years in secondary school education and another four years working with preservice teachers, I can attest that few ideas presented here are new or revolutionary. This comment, however, is not meant as a condemnation. On the contrary, I found the book's substance and presentation to be compelling and thought provoking. The more I read, the more I nodded in agreement with suggestions for effective questioning, setting personal goals, the use of formative assessment, scaffolding and spiraling instruction, and the importance of being genuine with students. With no conscious intent to do so, I found myself making numerous notations in the margins, my interest or imagination piqued by a specific example or idea. As I underlined noteworthy ideas and comments to discuss with my preservice teachers, I quickly realized that I was underlining a significant portion of the text.

Although written specifically for teachers working in urban schools, this book presents key characteristics of effective teachers and quality instruction in any setting. My students who are concentrating in secondary school mathematics education will be reading and discussing this textbook next semester.

—Steve Soltys
Elizabethtown College
Elizabethtown, PA

A Mathematical Tapestry: Demonstrating the Beautiful Unity of Mathematics, Peter Hilton and Jean Pedersen, 2010. 306 pp., $33.99 paper. ISBN 978-0-521-12821-6. Cambridge University Press; www.cambridge.org.



This culminating work of these two mathematicians is essential reading. Peter Hilton (1923–2010) worked with colleagues such as Turing and Feynman in his youth; Jean Pederson, a talented geometer in her own right, once collaborated weekly with George Pólya to further her thinking about polyhedra. This background contributes to the rich texture of *A Mathematical Tapestry*, which reveals how a seemingly simple paper-folding technique can lead to some of the most powerful ideas in mathematics.

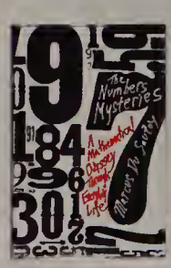
What I find most compelling about the book are its accessible tone and its mathematical complexity as well as the detailed connections made across so many different strands. The authors identify nine: paper folding, number theory, polyhedra, geometry, algebra, combinatorics, symmetry, group theory, and history. However, within each strand lie deeper ties to people and topics spanning centuries of mathematical thinking.

Hilton and Pedersen present a rare opportunity to embark on a true mathematical exploration. Building the models (I tried a few) definitely facilitates understanding, but readers with a strong mathematics background can follow the progression of ideas without creating each one. Make no mistake, however: This book involves serious mathematics. I especially appreciated the tips for untwisting flexagons and interpreting diagrams.

The authors recommend the book for "bright high school students." The more likely audience would be preservice secondary school teachers studying geometry or an ambitious mathematics department interested in a book study.

—Kasi C. Allen
Lewis and Clark College
Portland, OR

The Numbers Mysteries: A Mathematical Odyssey through Everyday Life, Marcus du Sautoy, 2011. 272 pp., $18.00 paper. ISBN 978-0-230-11384-8. Palgrave Macmillan; www.palgrave-usa.com.



Offering a million dollars to readers able to solve particular mathematical problems is certainly a unique way to begin a book. At the least, it leads readers to recognize that some seemingly simple mathematics has yet to be conquered. Technocentric readers will appreciate the QR codes,

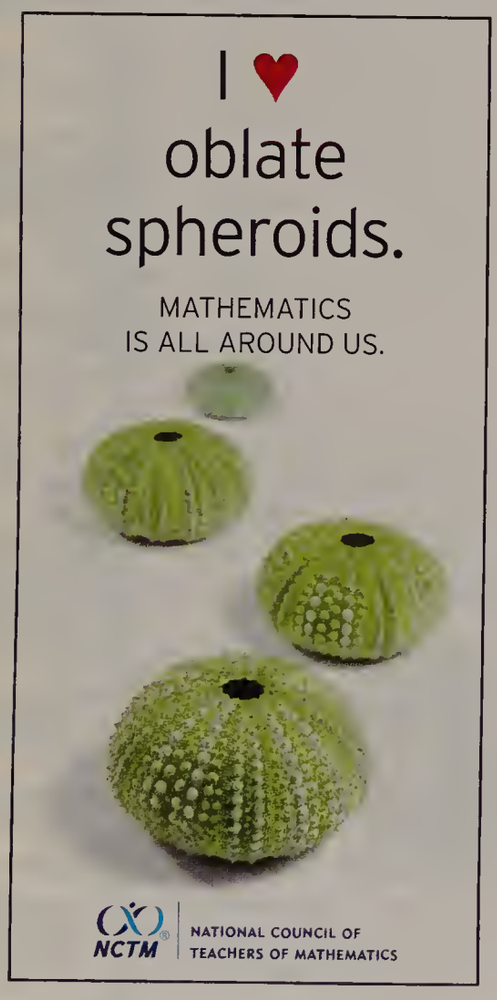
Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

which direct their smartphones and tablet PCs to rich Internet resources accompanying the text.

From investigations of prime numbers to geometry, game theory, combinatorics, and cryptanalysis, du Sautoy brings readers on a journey from the elemental, through a myriad of serendipitously interconnected topics, to the esoteric. Each chapter is captivating in and of itself, drawing readers in to follow each theme to the extent of their mathematical ability. However, du Sautoy weaves a serpentine topical path, demonstrating an impressive number of interconnections and applications, engrossing readers in mathematical investigations far beyond expectation.

Much of *The Numbers Mysteries* is beyond the rudimentary. High school and college mathophiles will benefit from this book as they recognize the intricately woven mathematical, scientific, historic, and sociological tapestry exposed by the authors. *The Numbers Mysteries* could be a valuable asset in further igniting the interests of fledgling mathematicians, as topics for further investigation or as a pleasant recreational read.

—Michael J. Bossé
East Carolina University
Greenville, NC



Rediscovering Mathematics: You Do the Math, Shai Simonson, 2011. 207 pp., $64.95 cloth. ISBN 978-0-88385-770-0. Mathematical Association of America; www.maa.org.

What exactly does a mathematician do? If mathematics is so challenging, why bother? These are the types of questions that Shai Simonson, with thirty years of experience in teaching many subjects, explores. Simonson effectively demonstrates why mathematics can be thrilling to teachers and students and yet so complex at the same time.

Through thirteen digestible chapters, Simonson travels the landscape of mathematics, visiting such topics as proof, series, figurate numbers, and game theory. Each chapter begins with an accessible observation, such as, Why do decimal expansions of some fractions terminate, whereas others repeat a particular cyclical length? Thematic topics are further explored, often by interspersing mathematical history or inviting puzzles. In his many examples, the author draws rich connections between algebraic, geometric, and numerical representations. Through transitional problems or open extension questions of varying difficulty, readers are frequently invited to engage in mathematical dialogue.

The book would be a captivating resource for two main audiences: teachers and gifted students. First, the author passionately advocates that teachers represent mathematics as an interactive enterprise that creates inquisitive students rather than memorizers. The problems and explanations serve as exemplary resources for teachers to act as "a conductor of a symphony," always probing students to dig deeper into generalizations. Second, the book could serve as a supplementary textbook for gifted students driven to explore nontraditional mathematical topics. The prose and detailed explanations make this book a must-have for the mathematically curious.

—Chris Bolognese
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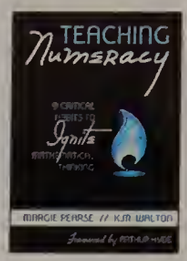
Teaching Numeracy: 9 Critical Habits to Ignite Mathematical Thinking, Margie Pearse and K. M. Walton, 2011. 240 pp., $33.95 paper. ISBN 978-1-4129-9223-7. Corwin; www.corwin.com.



"Help us revolutionize American mathematics instruction—one classroom at a time" (p. xii); this is the gauntlet cast by Pearse and Walton. The reason American students do not measure up in international comparisons of mathematics achievement, the authors posit, is because of their weak number sense. In addressing this weakness, Pearse and Walton have organized ten years of research experience in the field of numeracy into a framework for developing mathematically literate students—students who think through, rather than simply know, mathematics.

Part 1 provides a description of the nine habits that constitute this framework. The authors not only justify each habit in light of results from research but also provide concrete lesson ideas that they have used in their classrooms to help develop each habit. Lesson ideas ("How can I use this in my math class tomorrow?") are categorized by early elementary school and combined upper elementary and middle school levels, making this book a valuable resource for both practicing and preservice teachers at these levels.

In part 2, the authors propose five facets of numeracy-based lesson planning. Again, results from research provide a backdrop for each component, and the authors include more concrete examples for implementation. Appendices include three samples of numeracy-based lesson plans (division, elapsed time, and surface area of a right rectangular prism) that incorporate the five essential components as well as reproducible masters for implementation.

—Christopher Reisch
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- Aim for an informal style that appeals to *MT* readers (review recent issues of the journal for examples).
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- Proofread your manuscript in advance for English usage, mathematical correctness, and accuracy of references. Use a spell checker before submitting your manuscript and recheck it for grammar, completeness, spelling of all proper names, and full identification of all organizations and groups named by initials or acronyms.

TECHNICAL CONSIDERATIONS

Preparation of Manuscripts

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2. Put the manuscript title and name(s) of author(s) on a separate cover sheet. No author identification should appear in the manuscript. Do not reference your own work in a way that compromises the blind-review process.
3. Provide accurate and complete bibliographical information. All references cited in the manuscript should be listed at the end of the manuscript. See the *Chicago Manual of Style,* 16th ed. (Chicago: University of Chicago Press, 2010), for additional details.
4. Number all figures and tables; these may appear where appropriate in the text. Refer to them by number—for example, "see figure 1." *Include captions for all figures and tables; captions must be complete sentences.* The journal designer will prepare and position figures and artwork, so statements such as "insert figure 3 about here" are unnecessary.
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8. Use geometric symbolism and terminology appropriately. In particular, distinguish among segments, lines, rays, and angles and the measure of a segment or angle.
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All manuscripts must be submitted electronically to *Mathematics Teacher* at this site: **http://mt.msubmit.net**. An electronic acknowledgment will be sent on receipt of a manuscript.

MY FAVORITE lesson

Donna M. Young

A Graphic Organizer for Polynomial Functions

Students often view questions about polynomials—finding the zeros of a polynomial function, solving a polynomial equation, factoring a polynomial, or writing a polynomial function given certain properties—as discrete, unconnected processes. To address students' confusion about the many directions given for working with polynomial functions and to enable them to gain a true, conceptual understanding of polynomial functions, I created a graphic organizer (see **fig. 1**).

Using a graphic organizer to frame this content helps students (1) achieve the goals set forth by both NCTM and the Common Core State Standards Initiative and (2) recognize the same mathematical structure in different settings.

We begin with a circle in the center of the paper, which we label with the degree of the polynomial function under review. Surrounding this circle is an isosceles trapezoid, chosen because the two top vertices are closer to one another than the bottom two vertices. This proximity helps me accentuate that the directions "solve the equation $f(x) = 0$" and "state the zeros of $f(x)$" produce the same results. The two equivalent forms of a polynomial function—"general" and "factored"—are placed at the bottom vertices of the trapezoid.

A quadratic function illustrates that we can start from any vertex of the trapezoid and fill in the information for all vertices. For example, if we are given that the real and only zeros of the function are 2 and –3, we can create the graphic organizer in **figure 1**, beginning with the vertex at the upper right. If we start with $g(x) = 12x^2 - 19x + 5$, students can travel counterclockwise from the vertex at lower left.

Next, we focus on polynomials of degree greater than 2. We move from the general to the factored form, using factoring by grouping, the rational root theorem, and technology and discussing the appropriateness of each method. The factor and remainder theorems along with synthetic division allow us to move from the top vertices of the trapezoid to the bottom and vice versa.

When presented with a problem, students consider the organizer, locate the vertex that corresponds to the given information, determine which vertex corresponds to the required answer, and find a path to that answer. We work around the trapezoid, using only rational numbers at first; the result is some factored forms consisting of linear and quadratic factors.

We extend our work to include irrational zeros, allowing some factored forms to consist of all linear factors. When appropriate, we include complex zeros as well. Working with different number sets allows for differentiated instruction.

When mathematical procedures are complicated, the link to conceptual understanding can get lost, even when working with something as concrete as the roots of a polynomial function. Since creating and using this graphic organizer, my teaching of polynomial functions has become more coherent. The graphic organizer gives students a vehicle for traversing a complex set of procedures and multiple equivalent representations of a function. As a result, they better understand the connections between algebraic and graphical representations of polynomials, and they become more independent and confident learners.

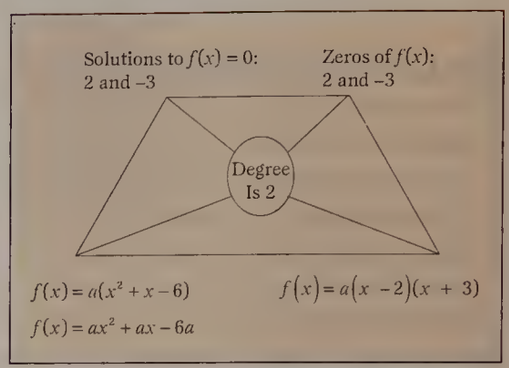

Fig. 1 The isosceles trapezoid is an ideal shape for the graphic organizer.

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The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by ***Jennifer Wexler,*** *wexlerj@newtrier.k12.il.us, New Trier High School, Winnetka, IL*

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